DIXIT algorizmi

[Thus Spake al-Khwārizmī]

Probably a 12th-century translation of text by al-Khwārizmī (d.ca. 850)

DIXIT algorizmi laudes deo rectori nostro atque defen-
sori dicamus dignas que et debitum ei reddant et aug-
endo multiplicent laudem deprecemurque eum ut nos
dirigat in semita rectitudinis et ducat in viam verita-
tis et nos adiuvet super bona voluntate in his que
decrevimus exponere ac patefacere de numero indo-
rum per IX literas quibus exposuerunt universum numerum suum causa
levitatis atque adbreviationis ut hoc opus scilicet redderetur levius querenti arith-
meticam id est numerum tam maximum quam exiguum et quicquid in eo est ex multip-
licatione et divisione collectione quoque ac dispersione et cetera.

Dixit algorizmi. Cum vidissem yndos constituisse IX literas in uni-
verso numero suo propter dispositionem suam quam posuerunt volui patefacere
de opere quod fit per eas aliquid quod esset levius discentibus si deus voluerit.
Si autem indi hoc voluerunt et intentio eorum cum novem literis fuit causa que mihi pa-
tuit deus direxit me ad hoc. Si vero alia de causa preter eam quam ego exposui
hoc fecerunt per hoc quod ego exposui eadem certissime et absque ulla dubitatio-
ne poterit inveniri. Levitasque patebit aspicientibus et discentibus.

Fecerunt ergo IX literas quarum figure sunt he. 9 8 7 6 5 4 3 2 1

Est quidem diversitas inter homines in figuris earum. Fit autem hic diversitas
in figura quinte et sexte. septime quoque et octave. Set in hoc nullum impedi-
mentum est. Sunt enim note signantes numerum et he sunt figure in quibus est illa
diversitas. 5 4 3 2 Et iam patefeci in libro algebr et almucabalah
id est restaurationis et oppositionis quod universus numerus sit compositus et quod uni-
versus numerus componatur super unum. Unum ergo invenitur in universo numero. Et hoc
est quod in alio libro arithmetice dixi. Quia unum est radix universi numeri et
est extra numerum. Radix numeri est quod per eum invenitur omnis numerus. Est autem unum [illegible]
quod invenitur per se id est absque alio aliquo numero [illegible] numerus autem sine uno inveni-
ri non potest. Cum enim [illegible] dicas inventione sui non indiget alio numero. Reliquus autem
numerus indiget uno. Quare non potest dicere duo vel tria nisi precedat unum.
Nichil aliud est ergo numerus nisi unitatum collectio. Et hoc quod diximus non potest
dicere duo vel tria nisi precedat unum. non de novo [illegible] ut indicabitur deinde. Non

Newsome added the numbers in the boxes.

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Dixit Algorizmi ('Thus spake Al-Khwarizmi') is the starting phrase of a manuscript in the University of Cambridge library, which is generally referred to by its 1857 title *Algoritmi de Numero Indorum*. It is attributed to the Adelard of Bath, who had also translated the astronomical tables in 1126. It is perhaps the closest to Al-Khwarizmi's own writings.

Thus Spake al-Khwārizmī: A Translation of the Text of Cambridge University Library Ms. Ii.vi.5

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The rules for manipulation of the Hindu–Arabic numerals 1, 2, 3, . . . and 0, otherwise known as algorism, became widely used in the West through Latin translations of Arabic from about the 12th century. Our principal Latin manuscript is Cambridge University Library Ms. Ii.vi.5. This manuscript has been published by Vogel and Yushkevich, but we present the first English translation. We have added a short introduction. © 1990 Academic Press, Inc.

Die Regeln für die Manipulation der indo-arabischen Ziffern 1, 2, 3, . . . und 0, auch Algorismus genannt, wurden von etwa dem zwölften Jahrhundert an vermittels lateinischer Übersetzungen aus dem Arabischen im Westen vielfach verwendet. Eines der bedeutenderen lateinischen Manuskripte ist Cambridge University Library Ms. Ii.vi.5. Eine Veröffentlichung des Manuskripts von Vogel und Yushkevich liegt allerdings schon vor, doch geben wir hier das erste Mal eine Übersetzung ins Englische und dazu eine kurze Einleitung. © 1990 Academic Press, Inc.

Le système pour l'utilisation des chiffres 1, 2, 3, . . . et 0, connu par ailleurs sous le nom d'algorisme, vit son emploi se développer largement en Occident à travers les traductions latines du texte arabe depuis environ le douzième siècle. Notre principal manuscrit latin est Cambridge University Library Ms. Ii.vi.5. Ce manuscrit a été publié par Vogel et Yushkevich mais nous présentons la première traduction anglaise. Nous y avons ajouté une courte introduction. © 1990 Academic Press, Inc.

AMS subject classifications: 01 A 30, 01 A 35, 01 A 75.
KEY WORDS: al-Khwārizmī, algorism, Hindu–Arabic numerals.

1. SYNOPSIS

The ordinary numerals we use today are the Hindu–Arabic numerals, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Originally these came from India and were slowly introduced into the West via the Arab world after the Hegira. The processes of calculation (addition, multiplication, etc.) using these numerals acquired the name of "algorism" because the texts on the processes were associated with the name of al-Khwārizmī.

At a recent conference, that "was a pilgrimage to the birthplace of algorithms" (D. E. Knuth, personal communication) and of al-Khwārizmī, it was pointed out by Knuth [1981] that a well-known Latin manuscript on algorism dating from the 13th century had not been translated into English (although a facsimile with notes had been published by Vogel [1963]). In this paper we present a translation of the

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text of this manuscript (Cambridge University Library Ms. Ii.vi.5) with an introduction and brief notes. We have found it necessary to add to our translation an introduction which attempts to clarify the status of the manuscript and of the text, since we found that there were a number of tortuous questions to be answered. Our conclusion, which is consistent with Vogel's notes and Yushkevich's views [1964a], is that this manuscript is a copy of an earlier Latin manuscript which was a translation or paraphrase of an Arabic manuscript, this latter being based on the work of al-Khwārizmī. There might well have been more intervening stages, for al-Khwārizmī lived about the first half of the ninth century and the manuscript dates from the 13th century.

2. AL-KHWĀRIZMĪ AND HINDU-ARABIC NUMERALS

The Hindu-Arabic numerals have their origin in India. The actual style of writing the numerals has changed significantly over the centuries and an extended account of their development can be found in Smith and Karpinski [1911]. It is well known that there was contact between India and the Arabs (see, e.g., [Holt et al. 1970, 483]). (In particular, one form of al-Khwarizmī's name includes al-Magusi which indicates a Zoroastrian connection, and the Zoroastrians were active in Khwarezm [Yushkevich 1964a, 186]). As far as the progress from India to the West is concerned, we have information that they were known and used in Syria in the seventh century. Nau, in his paper [1910 III], published part of a manuscript by Severus Sebokht (Paris Ms. Syriac, No. 346) "according to which the Indian numerals referred to as 'nine signs' were known and very properly appreciated in 662 by a Syrian in the Qenerez monastery" [Nau 1910, 226; present authors' translation] (see also [Sezgin 1974, 20]). al-Khwārizmī was alive less than two hundred years after this, and in tracing how the Hindu-Arabic numerals came to Europe all our references point in his direction. It is known that he was working in the early ninth century in the "House of Wisdom" (Bayt al-Ḥikma) of Caliph al-Maʿmūn in Baghdad. The House of Wisdom was the center for the translation of Greek and Latin texts into Arabic, and in the ninth century Arab science and learning became dominant around the Mediterranean as "Arab encyclopaedists . . . made it their business to transmogrify Graeco-Roman knowledge for their fellow countrymen" [Bolgar 1954, 166].

In the course of the Muslim conquests the Hindu-Arabic numerals and their arithmetic were carried by the Moors through North Africa and into Spain (cf. Lemay [1977]). Nevertheless they do not seem to have been used outside the Moorish lands before the middle of the 11th century. Now Arabic numerals were known to Gerbert in the 10th century, but he appears to have been unfamiliar with al-Khwārizmī's arithmetic, for apart from one work that is only doubtfully attributed to Gerbert "it seems now agreed that there is no direct influence of Arabian mathematics visible in Gerbert's writings." Moreover, Gerbert could surely not have already known all the algorism procedures using Hindu-Arabic numerals, including those for multiplication and division, since in 984 he "sent for the treatise of a certain Josephus the Wise on multiplication and division" [Haskins

1924, 8–9]. (For an excellent general account of this period see Murray [1978 II], and for the present context, pp. 173–174, in particular.) From 1050 onward Western science and mathematics did benefit from Arabic materials and their translators [Bolgar 1954, 170–171]. We shall return to the translators later.

The processes involved in using Hindu–Arabic numerals, as opposed to Roman numerals, acquired the name "algorism" in the West. It is generally agreed that the word "algorithm" comes from the name of the scholar Abū Jaʿfar Muḥammad ibn Mūsā al-Khwārizmī, who lived about 800–847 and used the Arabic language [Vernet 1978]. *The Oxford English Dictionary* states that the word "algorithm" was originally spelled "algorism" in its English version about the 12th century or perhaps slightly earlier [Simpson & Weiner 1989, 313]. It was only very much later that the word "algorithm," spelled with "th," became current. In fact, according to *The Oxford English Dictionary*, the first documented use in this sense in English is that in Hardy and Wright's *Introduction to the Theory of Numbers* [1938]. On the continent the variant with "th" was used in the 18th century [Knuth 1968 I, 2] with the modern, generalized meaning. In the 12th century and for a long time thereafter the spelling "algorism," with an "s," meant the rules and procedures for using the nine Hindu–Arabic numerals 1, 2, 3, 4, 5, 6, 7, 8, 9 and the cypher (Arabic "ṣifr") 0, though the actual shapes of these characters were different in those days. "Algorism" therefore referred only to a very small collection of algorithms in our modern sense (as used, for example, in mathematics and computer science). Nevertheless, those algorithms were the first in common use in the West, though of course Euclid had introduced some algorithms.

In this paper we shall use "algorithm" with "th" in the modern sense and "algorism" with an "s" when referring to the rules for calculating with Hindu–Arabic numerals.

The name of al-Khwārizmī, in its last part, reveals his family's origins. For "ibn Musa al-Khwārizmī" means "the son of Musa, of Khwārezm." The city of Khwārezm is now called Khiva and lies near the Aral Sea in the Uzbek Soviet Socialist Republic. In the time of al-Khwārizmī it was on important East–West trade routes. In 1979, as mentioned, it was the site of a conference celebrating the birthplace of algorithms and of al-Khwārizmī. The proceedings of that conference [Ershov and Knuth 1981] contain a paper [Zemanek 1981] that is a preliminary, but extensive, account of al-Khwārizmī and his times [1]. (There is also a thorough, although brief, account of al-Khwārizmī [Toomer 1973] in the *Dictionary of Scientific Biography*.)

We turn now to texts on algorism: the subject to which al-Khwārizmī gave his name. Vogel [1963] has identified four types of Latin text on algorism. All appear to derive from a lost Arabic original. However, there are sufficient differences between the types to assert that some are not direct translations from a single original text. From internal evidence we are sure that the manuscript that is our concern is not a direct translation from Arabic and, moreover, we believe the manuscript to have been (imperfectly) copied from an earlier Latin manuscript (see below, Section 3, and [Yushkevich 1964a, 187]). Nevertheless, it does appear that the text is derived from the work of al-Khwārizmī. This view is also shared by [Saidan 1978, 22]. By 1145 the *Algebra* of al-Khwārizmī had been translated into Latin by Robert of Chester [Sarton 1931, 126]; see also [Karpinski 1915]. The Arabic *Fihrist* of about 987 lists commentaries on the *Arithmetic* of al-Khwārizmī but neither the *Algebra* nor the *Arithmetic* is recorded. (See, for example, [Zemanek 1981; Saidan 1978].) However, works of astronomical interest that depend heavily on mathematics *are* listed [Karpinski 1915, 14; Lemay 1977, 446 (n. 47)]. It appears that *Joannis Hispalensis liber algorismi de practice arismetrice* (*John of Spain's book of algorism concerning practical arithmetic*, published by [Boncompagni 1857 II, 27–135]) is *not* a translation of al-Khwārizmī's work but "an adaptation made either by himself or by some Muslim author of al-Khwārizmī's *Arithmetic*" [Vernet 1978] (see also [Sarton 1934, 169]).

3. THE CAMBRIDGE UNIVERSITY LIBRARY MANUSCRIPT

Now we turn to the Cambridge University Library manuscript Ii.vi.5. The volume containing this manuscript—which belonged formerly to the monastery at Bury St. Edmunds, as evidenced by an ancient partial table of contents bound in with it—includes a number of other manuscripts, some of which are also mathematical. The volume is $5\frac{1}{2}$ in. by $7\frac{1}{2}$ in. and contains 125 leaves. It is of the 13th century, or possibly a little later, but no later than the 14th century [Allard 1987, 38], and the present manuscript occupies folios 104r–111v (previously numbered 102–109 as it says in the catalog [Cambridge 1858 III, 500–501]). Boncompagni [1857 I] first published this manuscript; Vogel [1963] published a facsimile with a transcription and notes. Yushkevich [1964b] published an article and a photographic reproduction of the Cambridge University Library manuscript and in his book [Yushkevich 1964a] he reproduced the *recto* of the first folio. Unfortunately all the reproductions are in black and white. The original text is written in a small hand in (now faded) black ink with the initials in a grayish blue, which does not appear to have faded much, and with decoration in red, which again is still bright.

The manuscript consists of eight folios and, as can be seen in the translation below, contains rules for arithmetical operations using the Hindu–Arabic notation. It breaks off in the middle of the treatment of the multiplication of $3\frac{1}{2}$ by $8\frac{3}{11}$. There is no surprising mathematical content in the manuscript, though the mixture is a little surprising, and it contains material of the same general nature as the other contemporary works on algorism, including those of John of Sacrobosco and Alexander of Villa Dei. Of the text of this manuscript (Cambridge University Library Ms. Ii.vi.5), Knuth says, "It would surely be desirable to have a proper edition in English, so that more readers can appreciate its contents" [Knuth 1981, 5]. Knuth's descriptions are quite accurate. He says in particular: "The original Arabic version of al-Khwârizmî's small book on what he called the Hindu art of reckoning seems to have vanished. Essentially all we have is an incomplete 13th-century copy of what is probably a 12th-century translation from Arabic into Latin; the original Arabic may well have been considerably different" [Knuth 1981, 85] [2].

Zemanek states that we have only one (13th-century) Latin manuscript of al-Khwārizmī's *Arithmetic*, which is in fact Cambridge University Library Ms. Ii.vi.5, but he also says "[T]he earliest form we have is an incomplete Latin translation . . . in Oxford" [Zemanek 1981, 53, 29].

Our original sole intention was to translate the "unique manuscript" mentioned by Zemanek, but several problems quickly came to light. When the first-named author took his first glance at the Cambridge catalog of manuscripts [Cambridge 1858], it became clear that the word "unique" is, to say the least, misleading. The rest of this introduction will, we hope, clarify the situation and show the position and relevance of Cambridge University Library Ms. Ii.vi.5 among some of the many manuscripts based on work attributed to al-Khwārizmī. It *is* clear that this manuscript is the only known manuscript of this text [Yushkevich 1964a, 187].

The first important question is: What is this text? Two items suggest that it is, at least in part, a copy of an earlier Latin text. First, whereas f.105r contains the appropriate Hindu–Arabic numerals, on most of the other folios these are completely omitted. (In a few cases it is not clear whether the symbol used is the Roman numeral I or the Hindu–Arabic numeral 1.) For unexplained reasons the gaps left for the Hindu–Arabic numerals that should be present on the other folios are far too large, while the Hindu–Arabic numerals that are present do have appropriate spacing, a matter not noted by Yushkevich [1964a]. (D. E. Knuth [personal communication] has suggested that the inappropriate gap sizes may be due to this manuscript being copied from another manuscript that had appropriate gaps everywhere but whose writer was unable, for some reason, to finish putting in the numerals in color.)

Further support for the view that the author of the Latin text did not fully understand the Hindu–Arabic numeral system is provided by the remark on fol. 105r to put "XX in the second [place]" where he actually means to put II which, by virtue of its position, represents two tens, i.e., twenty.

Again, on f.107r there is a sudden jump in the content. In the middle of giving an example of subtraction (1144 − 144) the text continues with a discussion of halving using the sexagesimal system, but such fractions are not introduced until the bottom of f.109v. Moreover, halving is discussed again, albeit briefly, on f.111v.

Finally there are errors of substance; for example, if we multiply seconds (i.e., fractions that are multiples of $1/(60 \times 60)$ or $1/60^2$) by seconds then in fact we get fourths (i.e., fractions that are multiples of $1/(60 \times 60 \times 60 \times 60)$ or $1/60^4$), rather than thirds, as stated in the manuscript on f.110r.

We therefore conclude that the copyist (or copyists, for it is not entirely clear from the handwriting that there was only one copyist) was unfamiliar with writing Hindu–Arabic numerals, was working from an earlier Latin manuscript, and did not fully understand the arithmetic that was being copied.

The next question is whether the text is a translation. It begins "al-Khwārizmī said" (Dixit algorizmi) and this is repeated 10 lines later (f.104r., l.11). On the same page the manuscript refers to "the book on algebra and almuqabalah" (Et iam patefeci in libro: algebra & almucabalah, l.22): presumably al-Khwārizmī's

Algebra [Rosen 1831]. On the *verso* of that folio, l.5, we read "al-Khwārizmī said" (Inquid algorizmi). Finally, on f.107r, l.18, there is another reference to "the book [on algebra and almuqabalah]" (Etiam patefeci in libro).

The introduction is somewhat similar to that in Gerard of Cremona's translation of al-Khwārizmī's *Algebra* (see [Hughes 1986, 233]). First there is praise to God and then a description of the numerals and the basic notation. Finally, on f.104r the author gives an Aristotelian account of the generation of number (cf. Aristotle, *Metaphysics* 987b) and just such an account is also to be found in the beginning of the *Algebra*. On f.104r the manuscript says "I have revealed in the book of algebra and almuqabalah," so we may conclude that al-Khwārizmī's *Algebra* (see [Rosen 1831]) was written by the original author before the present text.

Our discussion so far has, in essence, posed two questions: Where did the material in this manuscript come from? and What is its relation to writings of al-Khwārizmī? We have come to the following conclusions: the present work is a copy of a Latin translation made from an earlier Arabic version (or versions) of an original *Arithmetic* by al-Khwārizmī and that both (or all) the Arabic versions are lost.

Recently there has been a significant increase in the discovery and publication of Arabic manuscripts in the Middle East, so it is possible that an original may ultimately be found and the question of the material in the Cambridge University Library manuscript resolved.

We agree with Vogel's conjecture [Vogel 1963, 43–44] that the text of the Cambridge University Library manuscript and of all the other manuscripts on arithmetic that we have mentioned all came, after one or more reworkings, from a single Arabic source. On the other hand, the Cambridge University Library manuscript itself is unique in the sense that there is no other copy or even minor variant of this text known. There are a number of other algorisms. Halliwell [1841] published the *Carmen de algorismo* (*Poem on algorism*) of Alexander de Villa Dei. Many manuscript copies of that work exist and indeed it was one of the two most widely used texts of its period [Hughes 1980, 213; Benedict 1914, 126].

The other popular algorism text was written by John of Sacrobosco, who died in 1256. It is called *Algorismus vulgaris* (*Common algorism*) or *De arte numerandi* (*On the art of numeration*). A number of manuscripts of this work are extant and it was printed many times, including once in Cracow [Joannes de Sacrobosco 1521].

In addition to the manuscripts of works of John of Sacrobosco and Alexander de Villa Dei, we should perhaps mention two other Latin manuscripts on algorism. These were published by Karpinski [1921] together with an English translation of one of them. They are very similar to each other, but different from the other manuscripts we have discussed. One of them begins "Intencio algarismi est in hoc opere . . ." (The intention of al-Khwārizmī in this work is . . .) and the other is quite close in its wording. Both of these manuscripts are from the 12th century and therefore older than the Cambridge University Library manuscript Ii.vi.5. (Karpinski also mentions a French and an Icelandic manuscript, both of the 13th century.)

English translations of the two popular texts of Alexander de Villa Dei and John of Sacrobosco were published by Steele as *The Earliest Arithmetics in English* and date from the 15th century [Steele 1922, 4]. The *Carmen de algorismo* became *The Crafte of Nombrynge* and *De arte numerandi* became *The Art of Nombryng* [Steele 1922, 3–32, 33–51]. Halliwell [1841] had earlier published part of *The Crafte of Nombrynge*. Thus our manuscript and text alone have remained untranslated until now [3].

In conclusion we can surely claim that the Cambridge University Library manuscript is the unique manuscript of its text and that it is from the 13th century, although there are earlier, 12th-century, Latin manuscripts on algorism. We claim somewhat less surely that it is a copy of an earlier Latin text based on a Muslim version of a lost *Arithmetic* written by al-Khwārizmī.

4. THE TRANSLATION

The content of Cambridge University Library Ms. Ii.vi.5 is described in detail by Yushkevich [1964a, 1964b]. Here we give only a brief guide and in the notes we give a more detailed description. The content is generally straightforward and roughly follows the lines of other manuscripts on algorism. It starts with the notation system using the nine symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 and the cypher 0 (f.104r). Next comes the significance of the decimal places (f.105r). Addition and subtraction follow (f.106r). These are carried out from left to right in the same way as in the old Chinese standard style, rather than the modern Western right to left method [Saidan 1978; Lǐ Yǎn and Dù Shírán 1987] [4]. (It would be interesting to know whether there are connections between these Arabic and Chinese methods.) The treatment of subtraction is interrupted (see above) and a short section on halving intrudes (f.107r). The text then continues with multiplication. On f.108r the checking procedure by casting out nines is described and then division is discussed.

Fractions are treated beginning at the very bottom of f.110r. First it is pointed out that the Indians used a sexagesimal system. Next the use of this system is described. Doubling and halving receive brief mention on f.111v. Sezgin [1974, 239] suggests these processes may have an Egyptian origin (cf. [Peet 1923]), but although halving and doubling are found in Egyptian arithmetic, the work here is carried out in the Indian sexagesimal system, which was not used in Egypt. (Yushkevich [1964a], though mentioning the Egyptian tradition, does not go so far.) After this a discussion of other fractions begins. This breaks off abruptly at the end of f.111v in the middle of the computation of $3\frac{1}{2}$ multiplied by $8\frac{3}{11}$.

That the text is not a direct translation from the Arabic is supported by the inclusion of glosses. Thus, on fol. 104v, "But now let us return to the book" and, on fol. 105r, "But let us return to the book" surely mark the end of glosses.

The relationship between abacus calculations and the algorithms in the text of the manuscript is not clear. Further research is needed in this area.

We have added a short Appendix listing important corrections to both the manuscript and Vogel's transcription.

The following symbols are employed:

square brackets [...] enclose folio-references or indicate notes;
angled brackets ⟨...⟩ enclose material accidentally omitted by the copyist;
hooked brackets {...} enclose words mistakenly added by the copyist;
round brackets (...) enclose material of an explanatory nature supplied by the translators.

Most of the technical terms are readily translated but we have used "place" for "differentia" because this is what we would use nowadays, given the author's description.

One final question remains: Why was this manuscript not previously translated into English? One obvious answer would appear to be that other algorism works came into common use, such as those of John of Sacrobosco and of Alexander of Villa Dei, and some of these were translated into English long ago [Steele 1923]. However, the spread of algorism was relatively slow and there might have been a number of manuscripts like this one that were used only in a restricted circle and then superseded. Finally, it is clear that the copyist was not familiar with algorism and the misplacement of part of the text, plus the intrinsic difficulty of learning the new notation, might have made this particular manuscript less attractive.

5. TRANSLATION [5] OF CAMBRIDGE UNIVERSITY LIBRARY MS. Ii.VI.5

[Fol. 104r] Algorizmi said: Let us speak praises to God our guide and defender, worthy both to render Him His due and multiply His praise by increasing it, and let us entreat Him to guide us in the path of righteousness and lead us into the way of truth, and to help us in addition with goodwill in these things which we have decided to set out and reveal: concerning the numbering of the Indians by means of IX symbols (*literae*), by which they set out their universal system of numbering, for the sake of its ease and brevity, so that this work, to be sure, might be made easier for the seeker after arithmetic, i.e., the greatest number as much as the smallest, and whatever there is in it as a result of multiplication and division, also addition and subtraction, etc.

Algorizmi said: since I had seen that the Indians had set up IX symbols in their universal system of numbering, on account of the arrangement which they established, I wished to reveal, concerning the work that is done by means of them, something which might be easier for learners if God so willed. If, moreover, the Indians had this desire and their intention with these IX symbols was the reason which was apparent to me, God directed me to this. If, on the other hand, for some reason other than that which I have expounded, they did this by means of this which I have expounded, the same reason will most certainly and without any doubt be able to be found. And this will easily be clear to those who examine and learn.

So they made IX symbols, whose forms (*figurae*) are these: ⟨9 8 7 6 5 4 3 2 1⟩. There is also a variation among men in regard to their forms: this variation occurs in the form of the fifth symbol and the sixth, as well as the seventh and the eighth. But there is no impediment here. For these are marks indicating a number and the following are the forms in which there is that variation: ⟨5 4 3 2⟩. And already I have revealed in the book of algebra and almuqabalah, i.e., restoration and opposition [Rosen 1831], that every number is composite and that every number is put together above one. Therefore one is found in every number and this is what is said in another book of arithmetic [6]. Because one is the root of all number and is outside number. It is the root of number because every number is found by it. But it is outside number because it is found by itself, i.e., without any other number. But the rest of number cannot be found without one. For when you say one, because it is found from itself, it does not need another number. But the rest of number needs {needs} one, because you cannot say two or three unless one comes first. Number is therefore nothing else but a collection of ones, and as we said, you cannot say two or three unless one precedes; we have not spoken about a word, so to speak, but about an object. For [Fol. 104v] two or three cannot exist, if one is removed. But one can exist without second or third. Therefore two is nothing but the doubling or repetition of one; and likewise three is nothing but the tripling of this same unity; in this way understand about the rest of number. But now let us return to the book.

I have found, said Algorizmi, that everything that can be expressed in terms of number is also whatever is greater than one up to IX, i.e., what is between IX and one, i.e., one is doubled and two results, and likewise one is tripled and three results; and so on for the rest up to IX. § Then X is put in the place of one and X is doubled and tripled, just as was done in the case of one; from its doubling results XX, from its tripling XXX, and likewise up to XC. § After this C (a hundred) comes back in the place of one and is doubled there and tripled, just as was done in the case of one and X; and there will be produced from it CC and CCC etc. up to DCCCC (nine hundred). Again, a thousand is put in the place of one, and by doubling and tripling, as we have said, there result from it II thousand and III thousand etc. up to infinity according to this method. And I have found that the Indians worked according to these places. Of these, the first is the place of the units, in which is doubled and tripled whatever is between one and IX. The second is the place of the tens, in which is doubled or tripled whatever is from X to ninety. The third is the place of the hundreds, in which is doubled and tripled whatever is from C to DCCCC. Furthermore, the fourth is the place of the thousands, in which is doubled and tripled whatever is from a thousand to IX M. The fifth place is $\overline{\mathrm{X}}$ (ten thousand) [7] in the following way: every time the number rises, places are added. The arrangement of a number will be as follows: everything that will have been one in the higher place will be X in the lower, which is before it, and what will have been X in the lower will be one in the higher, which precedes it; and the beginning of the places will be on the right of the writer, and this will be the first of them and is itself placed there for the units. But when X was put in the place of one and was made in the second place, and its form was the form of one, they needed a form for the tens because of the fact that it was similar to the form of one, so that they might know by means of it that it was X. So they put one space in front of it and put in it a little circle like the letter o, so that by means of this they might know that the place of the units was empty and that no number was in it except the little circle, which we have said occupied it, and (sc. thus) it is shown that the number that is in the following place was a ten and that this was the second space, which is the place of the tens. And they put after the circle in the aforesaid second place whatever they wished from the number of tens from what is between X and XC and these are the forms of the tens: the form of X is thus ⟨10⟩, the form of XX ⟨20⟩. And likewise the form of XXX is thus ⟨30⟩, [Fol. 105r] and so on up to IX (sc. tens) there will be, clearly, a circle in the first place and a character pertaining to the number itself in the second place. Moreover, one must know this, that the character that signifies one in the first place, in the second signifies X, in the third C and in the fourth $\bar{I}$. And likewise the character that in the first place signifies two, in the second signifies XX and in the third CC and in the fourth $\overline{II}$ and understand likewise about the rest. But let us return to the book.

After the place of the tens follows {follows} the place of the hundreds in which is doubled and tripled whatever is from C to DCCCC and its form is just as the form of one put in the third place, thus 100, and the form of two hundred is just as the form of two placed likewise in the third place, thus 200; also the form of three hundred is the form of three placed in the third place, thus 300, and so on up to nine hundred. This place also is followed by the place of the thousands, in which likewise is doubled and tripled whatever is from a thousand to $\overline{IX}$ (nine thousand). The form of this is just as the form of one put in the fourth place, thus 1000; the form of two thousand is just as the form of two placed in the fourth place, thus 2000, and so on up to IX thousand; moreover, there are placed before [8] the character in the fourth place three circles, so that it may be shown what is in the fourth place, just as there were placed (sc. before the character) in the second place one circle and (sc. before the character) in the third place two circles, so that it might be shown what were the places of the tens and hundreds, and this happens when there is not before the number itself another number in the same (i.e., that) place. But if, along with the number that is put in these places, there is another number below it, it must be put in that place which is due to it. E.g., if there is along with X some number from those that are below [9] it, say as in XI or XII, they are placed thus 11 ⟨or 12⟩; i.e., in the first place, where the circle was placed, a one is to be placed and in the second place a one also is to be placed which signifies X. Likewise, if there is along with C another number from those that are below it, it is to be placed in the place which is due to it. Let us show this by a particular example and let us say that the number was: CCCXXV. When we wanted to put it in its places, we put it as follows: we began from the right of the writer and placed V in the first place and XX in the second going toward the left of the writer, and CCC in the third place, each number in its own place, i.e., the units in the place of the units, which is the first, and the tens in the place of the tens, which is the second, and indeed the hundreds in the place of the hundreds, which is the third, and this is the form 325; and it will be likewise in the other places according to this order, i.e., as often as a number is made larger and the places increase, each kind of number is to be put in its own place that is due to it. But when X or more is gathered in any of the places, it is to be raised to a higher place and from each X a one is to be produced in the higher place [10]. Again, if there is another number in the same place, at which a number arrives by increasing, it is to be added on and they are to be added together and if there is in it X or more, from each X a one is to be made and to be raised to a higher place, i.e., if ten is gathered in the first place [Fol. 105v], one is to be made from it and placed in the second place, and if in the same place there is likewise a number, it is to be added to it; and if there is X there, one is to be made from it and raised also to the third place. E.g., if in the first place, which is the place of units, you have X, make a one from it and place it in the second place. Moreover, in the first place put a circle just as we have said, so that it may be shown that there are two places [11]. But if there is XI, make a one from the X and put it in the second place as above and send down one into the first. But if you find some number in the second place, where you have placed the very number that you made from X, add it to that. And if there is X, or more, make from X a one and again place it in the third place; and what remains below X, let it remain in its own place. Moreover, what we say of more than ten holds for any large number. E.g., if there is in the second or third place a large number, such as if you find IX in the third place, which is the place of the hundreds, and if there is a X in the second place, make from the X a one and change it to the third place, and there add it to IX and there results X; make a one from the X itself and change it to the fourth place and there it will be a thousand. If, on the other hand, you found XX in the second place you would also make two from it [12], and adding two to IX in the third place, XI would also result; you would again make a one from the X and change it to the fourth place where it would be a thousand; and there remains a one in the third place and therefore indicates X or more. And this must be known that, because you have changed your number and put it in the following place, you must put it by means of its own characters, i.e., if it is X, instead of it place the character that signifies one in the first place, and if it is XX, instead of it place the character that signifies two in the first place. And understand likewise for the rest. But if there remains in the same place, from which you have changed a number, something from the number, move it down likewise by means of its own characters, i.e., if there remains a one or two, move it down there by the character that signifies the same number, i.e., if there remains one, copy there the character of one, and if there remains two, copy there the character of two, etc. But each form will have significance according to the place, i.e., in the first place it will signify units, in the second tens, in the third, hundreds etc., just as has been said above.

Moreover, if it is a large number and you wish to know which it is in numerical order or how many places are in it, so that you may write it in a book or talk about it, know that there is not in any place more than IX nor less than one unless there is a circle (i.e., o), which is nothing; when therefore you wish to know this, count the places beginning from the first, which will be on the right side, and this will be the place of the units. The rest of the places will be marked out by their succession toward the left side of the writer. Of these the second will be the place of the tens and the third of the hundreds and the fourth of the thousands and the fifth of the X thousands. Moreover, the sixth will be the place of the C thousands and the seventh of the thousand thousands. Again, the VIIIth will be the place of the X thousand thousands [Fol. 106r] and the ninth of the C thousand thousands, and the tenth of the thousand thousand thousands in three stages and the eleventh of the X thousand thousand thousands in three stages, and the twelfth of the C thousand thousand thousands in three stages and the XIIIth of the thousand thousand thousand thousands in four stages, and likewise in every place you will add according to the places of the number in your utterance. But if beyond three [13] places, i.e., the places of the hundreds and tens and thousands there is a one left, there will be X thousand of the thousands themselves that have resulted for you in words. But if there remain two places there will be C thousand of the thousands themselves. And now I have put together an example for you, by which you will be able to know and prove by it whatever is added to a number or subtracted from it: and this is the form of the same: ⟨1 180 703 051 492 863⟩.

When you add two symbols according to what we have said about these signs, the number of thousands of those signs will be one thousand thousand thousand ⟨thousand⟩ of thousands in five stages according to the number of characters that are below [14] them, and one hundred thousand thousand thousand of thousands in four stages according to the number of characters that are below them, and eighty thousand thousand thousand of thousands in four stages according to what is from those characters. Next seven hundred thousand thousand of thousands {of thousands} in three stages according to the characters that are below them and three thousand thousand of thousands in three stages and fifty-one thousand of thousands in two stages, and four hundred thousand and ninety-two thousand and eight hundred and sixty-three.

When you wish to add a number to a number or to subtract a number from a number, place both numbers in two rows, one of them, that is, below the other, and let the place of the units (sc. in one number) be beneath the place of the units (sc. in the other) and the place of the tens beneath the place of the tens. But if you wish to aggregate the two numbers, i.e., to add one to the other, you will add each place to the place that is above it with regard to its own kind, i.e., units to units and tens to tens. When ten has been collected in one of the places, i.e., in the place of the units or tens or in some other place, put a one instead of it and elevate it to a higher place, i.e., if you have ten in the first place which is the place of the units, make a one of it and raise it to the place of the tens and there it will signify ten. But if there remains something from the number that is less than X or the number itself is less than X, leave it in the same place. And if nothing remains, put a circle (i.e., o), so that the place may not be empty; but let there be a circle in it that occupies it, lest perchance, since [Fol. 106v] it is empty, the places may be decreased and it may be thought that the second is the first, and so you will be misled in your number. Do likewise also in all the places. Likewise, when X has been collected in the second place, make a one from it and raise it to the third place; and there it will signify one hundred and whatever is left over below X will remain there (i.e., in the second place). But if nothing remains in the other places, you will put a circle there as above. Do likewise in the rest of the places if there is more. § But if you wish to subtract a number from another, i.e., a number from a number, subtract each place from the other place that is above it according to its own type as has been said above. But if in the upper place there is not a large enough number from which you can subtract the number in the lower place, i.e., if it is less or nothing is there, take one from the second place, that is higher than that upper place, and from it make X, and subtract from it what you ought to and what remains leave in the same upper place. But if nothing remains, place a circle there as above. But if in the second place there is a zero from the upper place, take one from the third place and there will be X in the second. And again take one from that X and do with it as above and there will remain IX in the second place. And always begin in adding or subtracting from the higher [15] place; next from the following one that comes after it, because your task will be more useful and easier if God so wills. That this may be more easily understood, it is necessary that we note this under an example, and let us describe this in three [16] ways, so that no one may be confused in it in any way. Therefore let us take some number and say for example [17]: let us put six thousand four hundred and twenty-two in its places and say that we wish to subtract from it three thousand two hundred and eleven; so let us put two in the first place, that is on the right, and in the second XX. In the third also four hundred and in the fourth six thousand, and let us also put the very number that we wish to subtract from it, under it in the corresponding places, thus: i.e., let us place one under two in the first place, and X under XX in the second, two hundred also under four hundred in the third, and three thousand under VI thousand in the fourth and this is their form:

$$\left\langle \begin{matrix} 6422 \\ 3211 \end{matrix} \right\rangle$$

Since we wanted to subtract one number from the other, i.e., the lesser from the greater, we began from the higher place, i.e., from the fourth. So we took III from VI and there remained three in the fourth place. We also took two from IIII and there remained two in the third place. We also took one from II and there remained one in the second place and likewise there remained one in the first place, when we took one from two that was above it, and this [Fol. 107r] is the form of the remainder: ⟨3211⟩. Again let us put another number in another way as you like, so that nothing may remain from it in its places. Let our number be one thousand one hundred and forty-four, from which let us take CXLIIII and let us place each of them under the other in this way [18]:

$$\left\langle \begin{matrix} 1144 \\ 144 \end{matrix} \right\rangle$$

When you wish to halve some number, begin [19] from the first place and halve it: if there is an uneven number in it, halve the evens and there will remain one, halve this, i.e., divide it into two halves, and make one half thirty parts out of sixty, that make one, and place XXX under the same place; then halve the following place if its number is even; and if it is uneven, take the half of the even number and put it in its place and make the half of the remaining one five and place it in the place that is before it. But if there is nothing but one in the same place that you wish to halve, put in its place a circle and put five in the place that is before it. And work likewise in all the places. And when you want to double, begin from the higher place and double, and when the number by increasing has exceeded X, make one out of the ten and put it in the following place and you will succeed, if God so wills.

I have also set forth in a book what is necessary for every number that is multiplied by some other number, so that one of them may be multiplied according to the units of the other. When you want to multiply some number by another using the Indians' symbols, you must remember the multiplication of number that is between one and IX in turn, whether the number is the same or different. When you want to multiply a number by another, place one of them according to the extent of its places on a tablet or whatever else you like. Then put the first place of the one number under the higher [20] place of the first. The first place of the same number will be under the last place of the first number that we put down. And the second place will precede the first number toward the left. An example of this is [21]: when we wished to multiply two thousand three hundred XXVI by CCXIIII, we put two thousand three hundred XXVI by means of Indian symbols into IIII places, and in the first place, that is on the right, there was VI, and in the second two, that is XX, and in the third three, that is three hundred, and in the fourth two, that is two thousand. After this we placed under two thousand IIII, then in the preceding place toward the left one, which is X, then in the third two, which is two hundred, and this is their form:

$$\begin{pmatrix} & 2326 \\ 214 & \end{pmatrix}$$

[Fol. 107v] After this begin from the last place above and multiply it by the last place of the lower number, which is under it. And what results from the multiplication, write up above. Next write also in the place that comes next by returning toward the right of the lower number. Then do likewise, until you multiply the last place of the upper number by all the places of the lower number. And when you have completed this, transfer the lower number one place toward the right. And the first place of the lower number will be under the place that comes after the number that you have multiplied toward the right. Then put the rest of the places successively; after this also multiply the number itself, under which you put the first place of the lower number, by the last place of the lower number; then by that which comes next, until you have done them all, just as you did in the first place. And whatever is accumulated from the multiplication of each place, write it in the place above it. And when you have done this, transfer also that number, that is your own, by one place and do with it just as you did in the first places. And do not cease from so doing, until you complete all the places. And thus multiply the whole upper number by the whole lower number. And when it happens that the first place of the lower number is under some place in which there is no number, i.e., in which there is a circle, let us make it go across to the following place in which there is a number toward the right. Because every circle that is multiplied by some number is nothing, that is to say no number arises from it and whatever is multiplied by a circle is likewise nothing. And when we have transferred the places toward the right and afterward have multiplied the upper number itself by each place from the lower number, we shall add what results to us from the multiplication above the place that is above that place by which we have multiplied; and, provided that, as the number increases, X is accumulated for us in some place, we shall make one from it and place it in the following place toward the left. And if something remains, we shall indicate it in its own place; but if nothing is left over, we shall put a circle in its place, in case anything may be subtracted from the places, and when the multiplication has proceeded to the first place of the lower number, we shall delete whatever was in the place that is above it, and in its place we shall indicate what has resulted for us from the multiplication. And we shall so do until we multiply all the places of the upper number by all the places of the lower number. And thus from these we shall multiply [Fol. 108r] the number according to the number of units of the one and the multiplication will be completed. And this is the form of the number that resulted for us from the multiplication of two thousand three hundred and twenty-six by two hundred XIIII, that is four hundred thousand and ninety-seven thousand and seven hundred LXIIII ⟨497764⟩.

When you wish to know whether you have succeeded or made a mistake in your doubling or multiplication [22], take a number that you wish to double and divide it by IX and IX and whatever remains less than IX double it; if there is IX in this, discard it and keep the remainder. After this double your number, that is the number itself that you wish to double, and divide it by IX and IX and what remains, if it is the same as that which had first remained while you had doubled it, you are now correct: but if not, you have made a mistake; and when you want to multiply some number by some (sc. other number) and wish to test it as above, divide the number that you have doubled by IX and keep what remains below IX. Once more divide the other number by IX and keep what remains below IX. Then multiply what remains from the first by that which remains to you from the second, and from that which has been accumulated discard IX if it is there and if IX is not there, a symbol will remain. But if IX is there, discard IX and keep what remains and understand that this will be a symbol; after this multiply one multiplicand by the other and divide the product by IX, and if what remains is the same as that which I have told you about the symbol, know that you were right. But if it is not the same, understand you were wrong.

In division [23] on the other hand put the number that you wish to divide in its places; next put the number itself by which you wish to divide under it. And let the last place of the number by which you are dividing be under the last place of the upper number that you are dividing. Moreover, if the number that is the last place of the upper number that you wish to divide is less than that which is the last place of the lower number by which you are dividing, retract the place itself toward the right, until the number of the upper number is greater, i.e., put the last place of the lower number by which you are dividing under the second place that comes after the last place of the upper number. After this consider the first place of the number by which you wish to divide and put in its column above the upper number that you are dividing or beneath it in its column some [Fol. 108v] number which, when you have multiplied it by the last place of the lower number by which you are dividing, will be the same as that number that was in the upper place or close to it, provided it is less than it. When you know it, multiply it by the last place of the lower number and subtract what results to you from the multiplication from that which is above it, from the upper [24] number that is being divided. Once again multiply it by the second place that comes after the last place toward the right, and subtract it from that which is above it and proceed in the division just as you proceeded in the beginning of the book, when you wished to subtract some number from another number; and likewise proceed until you multiply it by all the places of the lower number by which you are dividing. After this move all the places of the lower number, by which you are dividing, one place toward the right. And put in the column of that first place (sc. a number) like that which you previously placed. When you have multiplied this by the last place of the lower number by which you are dividing, it [25] will cancel that which is above it or that was close to it, and multiply that which you put in its column by the last place of the lower number and subtract what results for you from the multiplication from that which is above it. And do likewise in all the places, and if there remains from the places of the upper number that you are dividing something that must be divided, always move the places of the lower number by one place, until its first place is in the column of some place of the upper number; but if there is a circle in some place among the places of the number that you are dividing and the moving has reached it, do not pass it, as you did in multiplication, but put in its column something which you will multiply just as we recounted. When you have completed all this, whatever has resulted for you from the places in the column of the number that you are dividing, that is owed to one and if anything remains, it will be a part of one from that number that you are dividing, and never will there remain anything except what will be less than that which you are dividing. If more remains, you may be sure that you have made a mistake.

And know that division is similar to multiplication, but this is done inversely, because in division we subtract and there we add, i.e., in multiplication is its model. But when we wanted to divide forty-six thousand and four hundred and sixty-eight by three hundred and XXIIII, we first put eight on the right side, then we put six toward the left which is sixty, then IIII which is four hundred, then six which is VI thousand, then IIII which is forty-thousand. Of these places the last toward the left will be IIII, and the first of them toward the right eight; [Fol. 109r] after this write under them the number by which you are dividing, and write the last place of the number by which you are dividing, which is the form of three and is three hundred, under the last place of the upper number which is IIII, provided that it is less than that which is above it: and if it were more than it, we shall move it by one place and put it under the six. After this we shall put in that place, that comes after the three, the form of two which is XX, beneath the six; then we shall put in that place, which is beneath the IIII, likewise IIII and this will be their form:

$$\left\langle\begin{matrix}46468\\324\ \ \ \end{matrix}\right\rangle$$

After this to begin [26] with let us write one in the column of the first place of the number by which you are dividing, above the upper number that you are dividing which is four. And if we had put it beneath the IIII, it would be equally appropriate. Let us multiply it (i.e., one) by three, and we shall subtract it (i.e., 3) from that which is above it, and there will remain one. Then let us multiply it by two and subtract it (i.e., 2) from that which is above it, which is VI, and there will remain IIII. After this let us multiply it again by IIII and subtract it (i.e., 4) from that which is above it, which is IIII and nothing will remain; and we shall put a circle in its place. Next move the beginning of the number by which you are dividing, i.e., IIII, beneath VI and there will be two beneath the circle and III beneath IIII. Then write in the column of the lower number something in the column of the one, i.e., IIII, and multiply it by three and there will be XII; and subtract it (i.e., 12) from that which is above three which is XIIII and there will remain II; after this multiply also IIII itself by the two that follows the three and there will be VIII, and subtract it (i.e., 8) from that which is above it, that is XX and there will remain XII, i.e., two above the II and one above the three. Again multiply IIII by IIII which comes next on the right, and there will be XVI; and subtract it (i.e., 16) from that which is above it, which is CXXVI and there will remain above the IIII a circle and above the two, one, and above the three, one. Again move the number by which you are dividing, i.e., IIII, beneath VIII, ⟨and⟩ there will be two beneath the circle and three beneath the one; next write three in the column of IIII above the upper number that you are dividing in the row of IIII and one (i.e., 143), multiply it (i.e., 3) by three, and there will be IX; and subtract it (i.e., 9) from that which is above three which is XI and there will remain two above the three. Multiply also the three by the two that follows the three and there will be VI, and subtract it (i.e., 6) from that which is above the three, which is XX; there will remain XIIII. Once more multiply the aforesaid three by IIII that follows the two and there will be XII, and subtract it from that which is above it, which is CXXVIII [27]; and there will remain six above the IIII, and three above the two, and one above the three. And there will come out for us what is owed to one from it and this will be CXLIII and CXXXVI parts from CCCXXIIII parts of one (i.e., $143\frac{136}{324}$). And this is their form:

$$\left\langle\begin{matrix}143\\136\\324\end{matrix}\right\rangle$$

[Fol. 109v] And if you wish to divide several places by one place, such as one thousand DCCC by IX, you will write one thousand eight hundred, whose form is that you put two circles on the right, next VIII, and then a one [28]. After this you put IX under the VIII, provided that it is more than VIII; then write in its column above the VIII something which when you multiply it by IX will cancel what is above it, i.e., XVIII, which is above the IX, and finding that to be two, multiply it IX, IX and there will be XVIII; subtract it (i.e., 18) from that which is above and there will be no remainder; then move the IX by one place toward the right and it will be beneath a circle. Put up above something which, when you multiply it by IX, will be zero; because there is a circle above the IX, and there is no number there. So put a circle in the column of IX in the row of the two and multiply IX by the circle and there will be a circle, i.e., zero. After this also move the IX to the place that is before it (i.e., to the right), which is the first place, and there will be IX beneath a circle and do with it just as you did with the circle, that was to the left [29]; and there will be two circles there, after which there will be a two, which is two hundred and this is what is owed to one and there will not remain anything from that which is being divided, and as often as you divide some number by some other number and there remain circles from that which is being divided before (i.e., to the right of) which there is no number, take what is left over from the circles from the beginning of the places of the divided number toward the right and add them to that which has resulted from the division and what there is is the very thing that is owed to one. And this is a certain approximate abbreviation. The first row is the row of work. An example of this is that when we wrote one thousand DCCC, there were two circles and VIII also in the third place and one in the fourth; we put the IX below the VIII because it is more than that which is in the last space; and its form was thus:

$$\left\langle \begin{array}{r} 200 \\ 1800 \\ 9 \end{array} \right\rangle$$

And when we wrote two in the column of IX above VIII, and multiplied it by IX, there were XVIII, and when we subtracted it (i.e., 18) from that which is above IX, there remained two circles with no number before (i.e., to the right of) them. So we wrote two circles in the row of the two that is above the IX and there was CC, whose form is this: ⟨200⟩.

This is everything that is necessary to men for division and multiplication in the case of a number that is whole. And now we shall begin to treat the multiplication of fractions and their division and the extraction of roots, if God so wills.

Know that fractions are called by many names innumerable [Fol. 110r] and infinite such as half, third, quarter, ninth and tenth and XIIIth and XVIIIth, etc. But the Indians constructed their fractions out of sixty: for they divided one into LX parts which they called minutes. Once more (sc. they divided) each minute into LX parts, which they called seconds; and one minute will be out of LX and one second out of three thousand and six hundred and each second is once more divided by LX, and a third will be out of two hundred and XVI thousand and each third is divided by LX fourths, and so on to infinity will be the places. Thus the first place of the degrees is the place in which stands a whole number, and in the second place will be the minutes. In the third also are the seconds and in the IVth the thirds and so on in the IXth and Xth place. And know that every whole number that is multiplied by a whole number yields a whole number, and every whole number multiplied by some fraction yields (sc. a product) according to the nature of that fraction; and two degrees multiplied by two minutes will be IIII minutes and three degrees by six thirds will be XVIII thirds. Minutes also multiplied by minutes will be seconds and seconds by seconds [30] will be thirds and thirds by thirds will be fourths and fourths by fourths will be fifths, because you join both places that you multiply in turn; and what is aggregated from the number of fractions is like that which results from a whole number multiplied in turn. For example [31]: six minutes multiplied by VII minutes will be XLII seconds, because minutes are parts out of LX parts of one integer and when you multiply parts out of LX by (sc. parts out of) LX there will be what results from the multiplication of LX by LX which is three thousand six hundred; and likewise VII seconds multiplied by IX minutes will be LX-three thirds; and all LX of these will also be one second and there will remain three thirds, because minutes are parts out of LX and seconds are parts out of three thousand and six hundred. Thus multiply them in turn and there will result parts of two hundred and XVI thousand which are thirds and are LX (each) out of three thousand six hundred.

And when you want to multiply [32] one and a half by two [33] and a half, make one and a half into minutes and there will be XC. Once more make the one and a half by which you wish to multiply into the same minutes and there will likewise be XC; multiply one of them by the other and there will be VIII thousand C seconds; divide the seconds by LX and there will be minutes, because [Fol. 110v] every LX IIs (i.e., seconds) make one minute. And there will result for you CXXXV minutes; and divide them by LX and there will be degrees, because every LX minutes make one degree. And this will be one integer from the number; and there will result for you two (sc. degrees) and XV minutes, which are one quarter of one.

And if you wish to multiply [34] two whole numbers, i.e., two degrees and XLV minutes, by three whole numbers and X minutes and XXX seconds, put the two whole numbers as minutes, i.e., multiply them by LX and there will be CXX, to which add the above-mentioned XLV minutes, and there will be CLXV minutes; keep them, because you have now rendered them into the lowest place. After this make the aforesaid three degrees into minutes by multiplying them by LX as above. To these add the X aforesaid minutes and there will be CXC minutes; then put the CXC minutes themselves into seconds by multiplying them once more by LX, until you render them into the last place, i.e., into seconds. There will be XI thousand four hundred, to which add the XXX seconds, that belong with them. And there will be XI thousand four hundred ⟨and thirty⟩ seconds. And so render them into the last kind of fraction of the same number. Multiply all the above-mentioned by CLXV minutes and there will be one million eight hundred and eighty-five thousand and nine hundred and fifty thirds, because you have multiplied them, i.e., the seconds, by minutes and they have become thirds. Divide these by LX so that they may be made into seconds. And there will result for you XXXI thousand and four hundred XXXII seconds and there will remain XXX thirds. Likewise, divide the seconds by LX so that they may be made into minutes. And there will result for you five hundred and XX-three minutes and in addition there will be LII seconds. Again divide the minutes so that they may be made into degrees, i.e., a whole number. And there will be VIII (i.e., degrees) and there will remain XLIII minutes. And everything that has resulted from the multiplication will be eight degrees and XLIII minutes and LII seconds and XXX thirds. And do likewise concerning all fractions, i.e., make each of them that you wish to multiply into another lower place, that will be common to them. After this multiply one of them by the other and keep what results and see from which of the places it is; then divide by LX, just as I told you, so that you may raise them to degrees, or, if you like, degrees will arrive from the places that are below and there will be precisely what resulted to you from the multiplication of one of them by the other. And there is for that another shorter method; but this is the arrangement that the Indians used to form their system of numbering.

Know that when you wish to divide a number with a fraction by some number with a fraction or a number with a fraction by a whole number or a whole number by a number with a fraction, you must make each number of the same nature, i.e., turn both numbers into the lower place. E.g., if the lower place is of seconds, put each number into seconds; but if there are thirds in one of them and seconds with the other, turn both into thirds, and if there is something with one of them from the fourth or sixth or something else lower than these places, while the other number is an integer, turn both into that place which is lower in both; then divide what you wish by what you wish, after you have made each number of one kind, and what results [Fol. 111r] will be degrees, i.e., a whole number, because in the case of any two numbers that are of one kind, if one of them is divided by the other, what results will be a whole number. E.g., [35] if XV thirds are divided by six thirds, there will result from the uniformity of the division two-and-a-half; because XV thirds make V wholes and when you divide them by VI thirds, which is two wholes, there will result two and a half. And likewise halves are divided by halves and fourths by fourths, minutes also by minutes {by minutes} and seconds by seconds and thirds by thirds. And [36] when you wish to divide X seconds by V minutes, make the minutes seconds, so that they are of one kind of one place; and there will be three hundred seconds; and as long as you wish to divide X seconds by them, X cannot be divided by three hundred. Know therefore that a whole did not result. So place a circle in the place of one and multiply X by LX and there will be six hundred, and when you divide this by three hundred, there will result two, which are two seconds [37]; and this is what is owed to one; because when you multiplied them (i.e., the ten seconds) by LX and then divided, you now reduced them by one place, which are seconds [38]. And know that as regards every number that is divided by another number, if what is extracted from that which is divided is multiplied by that by which it is being divided, the first number will return, i.e., the number is being divided. Of this an example is: that when you divide L by X, there will result what is owed to one, i.e., five. And when you multiply that which has resulted to you from the division, i.e., five, by that by which you are dividing, which is X, there will return the first number, i.e., L. When therefore we divided X seconds by V minutes, there resulted what is owed to one, i.e., two seconds [39]. And when we multiplied two seconds [40], i.e., what resulted to us from the division, by that by which we divided, which is V minutes, X seconds were produced and this is a proof of the division [41]. Likewise [42], when you wish to divide X minutes by V thirds, change the minutes into thirds, and there will be XXXVI thousand thirds; and divide by V thirds and there will be VII thousand two hundred degrees and this is what is owed to one. And when you wish to check this, multiply VII thousand two hundred degrees by V thirds and there will result XXXVI thousand, and when you divide this by LX there will result VI hundred seconds and when once more you divide VI hundred seconds, there will be ten minutes.

When you wish to set out a whole number and fractions, put the whole number in a higher place; then put whatever is from the first place, which are minutes, beneath the whole number and the seconds under the minutes and likewise the thirds under the seconds and the rest as you wish according to the places. Of this an example is: that when you wanted to set out XII degrees and XXX minutes, together with XLVI [43] seconds and L fourths, we set down XII. After this we put beneath it XXX in the place of the minutes and beneath XXX XLV in the place of the seconds. In the place of the thirds we put circles, because there was a lack of thirds, and so that we might know that there were still fourths remaining. Then we put under the circles fifty in the place of the fourths and this is their form [44]:

$$\begin{pmatrix}12\\30\\45\\00\\50\end{pmatrix}$$

And likewise we put all the places of the fractions under each other in turn. And as often as LX or more are collected in some place, we shall put in their position, i.e., in their place, whatever is left over above LX and we shall make one out of every LX. [Fol. 111v] We shall put it in the higher place. And likewise if we wish to subtract [45] fractions, we shall begin from the higher place and we shall subtract each place from that which is above it. But if there is in the same higher place less than that which you wish to subtract from it or if there is a circle in it, subtract one from the place that is above it, and the one itself will become LX parts from the fraction that you are working and subtract from it what you are working and add what is left above the incomplete place. And if there is a circle above the place itself, subtract one from the place that is above it, and render it into LX parts in the place that is below it. Then subtract one also once more from it and make it into parts as above in the place that you wish. After this subtract from it what you wish and what is left put in that place in which that which is subtracted from it is completed.

And when ⟨you wish⟩ to double some number or fraction, begin from the higher place, then from that place that comes after it. And when there has been collected in any of the places something more than the number of its parts, put the excess in that place and subtract one from the place which is above it. In halving also begin from the lower place and halve it, then the following place, and if you find one there, do with it just as I explained to you in the beginning of the book. § And if you wish to multiply fractions and a number and fractions apart from minutes or seconds, such as quarters and sevenths and the rest of the parts similar to these, and to divide them into each other, the work will be just as the work for minutes and seconds. And I shall set out an example for you if God so wills. And I have already explained to you in the multiplication of minutes and seconds and thirds concerning the two numbers that you wish to multiply together, i.e., one of them by the other, in what manner you will make them into one kind, so that you may change them, i.e., into the kind of the lowest place, i.e., if the lowest place is of seconds, change them into seconds, and if it is of thirds, change them into thirds, etc. Do likewise in the case of parts, i.e., if the last place is of fifths or of sevenths, make your number of the nature of the same part. After this multiply it in turn, and raise what results to a whole number, i.e., divide it by (sc. a number) like the same kind multiplied by the other kind, as if you wished to multiply III sevenths by IIII ninths, and those sevenths and ninths were in the first place of the fraction as if minutes; and you were multiplying them in turn and they were being made in their own place from the kind of seconds. And when you wish to raise them to a whole number, divide them by both places which are sevenths by ninths. But if something else is divided and results from the division, it will be a whole number, and if it cannot be divided, there will be parts of one of the same kind as those by which you divided. Three-sevenths by IIII-ninths will be XII parts from LX-three parts of one [46]. When therefore you wish to multiply three and a half by VIII and three XIths, write three and put it beneath one and two beneath one. And now you have written three and a half, because a half is one part of two just as one minute is one part of LX parts of one. After this write in another place VIII and beneath it three and beneath three XI.

$$\begin{pmatrix} 3 & 8 \\ 1 & 3 \\ 2 & 11 \end{pmatrix}$$

And thus set out VIII ⟨and III XIths . . .⟩.

APPENDIX: ALTERATIONS TO VOGEL'S TRANSCRIPTION

The following changes have been made by the translators in Vogel's transcription. Some of these are mere questions of punctuation, but others involve incorrect readings made by Vogel (and not picked up in Vogel's *Errata* sheet) and mistakes in the ms. text itself not identified by Vogel. This list is not intended to be exhaustive. (References are to the line numbering of the ms., and *not* to the present translation.)

Folio 104r

l.7 Print a comma after "suum" (the ms. indicates punctuation here).
l.16 Ms. reads "certissime," not "certissima."
l.22 The numbers referred to are clearly ⟨5 4 3 2⟩, not ⟨5 6 7 8⟩. (Vogel fn. 3 appears to sense that something is wrong with his supplement.)

Folio 104v

l.16 Read "decenorum," for "decenorum;".

Folio 105r

l.14 "unius" must be a scribal error for "unus" (sc. circulus).
l.24 The ms. reads ".XX. et in tercia."

Folio 105v

l.17 The ms. reads "in sequenti differencia."

Folio 106r

l.9 "due litere" (i.e., duae literae) affords a doubtful syntax ("Two symbols [nominative], when you add them . . . , the number will be . . ."), but the sense is clear and "two" (not "three," *pace* Vogel fn. 2) seems to be correct.
l.30 "dimittens" must be a scribal error for "dimittes."

Folio 106v

l.1 Vogel should not have altered the ms. reading here—"secunda esse prima" is quite correct: "differentia" is to be understood, and is the subject of "putetur."

Folio 107r

l.5 "accipe" may be a scribal error for "incipe."
l.9 Delete the punctuation after "sequentem differentiam." There is none indicated in the ms., and none needed.
l.17 Ms. reads "si," not "sic."
l.20 Ms. reads "volueris", not "voleris."
l.27 After "sinistram" insert a full stop, not a comma.
l.30 Ms. reads ".VI. et in secundo" (a mistake for "secunda").

Folio 108r

l.18 Delete punctuation after "nota" (none indicated, none needed).
l.21 Delete punctuation after "errasti" (none indicated, none needed).
l.32 Ms. reads "diuidis", not "diudis."

Folio 108v

l.5 The ms. erroneously reads "inferiori" instead of "superiori." (There is a certain amount of confusion in Vogel's fn. and *Erratum*).

l.13 Ms. erroneously reads "consument" (3rd person plural) instead of "consumet" (3rd person singular).

l.27 Read "e converso," not "econverso."

Folio 109v

l.12 Ms. reads "ad", not "at."

l.14 We doubt the impossible reading "eos," reported with understandable surprise by Vogel, but we cannot decipher the correct reading. The sense, however, is clear: he is talking about the first 0 after 8 in 1800.

l.22 Ms. reads "VIII et in tercia" instead of "VIII in tercia."

l.23 Ms. reads "sint," not "sunt" (although the latter may be the required reading).

Folio 110r

l.17 "coniunctis" must be a scribal error for "coniungis."

l.31 "quam" is a scribal error for "in quo."

Folio 110v

l.15 The ms. appears to read "idest in secundas in minutiis", but this should be "idest secunda in minutis."

l.25 There is no punctuation in the ms. between "ipsum" and "erit," but a comma is required.

Folio 111r

l.4 The ms. correctly reads a full stop after "dimidium."

l.8 The ms. correctly reads "quae," not "quem." After ".X. secunda" read a comma (not a semicolon).

l.20 Print "minu^{ta}" as "minuta."

Folio 111v

l.1 After "differentia" read a full stop (not a semicolon).

l.3 Ms. reads "ipsam," not "ipsa."

l.11 ⟨volueris⟩ does not appear in the ms. but must be supplied.

l.20 After "alio" read a comma (not a semicolon).

l.26 Ms. reads "IIII^{or}," not "III^{or}."

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NOTES

1. We should point out here that Zemanek is misleading in some of his comments in his article (see also below).

2. However, in the same volume, Zemanek writes, "The earliest form of al-Khwārizmī's arithmetic we have is an incomplete Latin translation, probably of a revised Arabic version, in Oxford, but the original can be detected in many later manuscripts" [1981, 29]. It is indeed true that there is a late 12th-century Latin manuscript on algorism in Oxford (Ms. Selden Supra 26), but there are also several others of about 1300. Later in the same paper, Zemanek writes, "We have only one manuscript of a Latin translation of al-Khwārizmī's arithmetic, probably written in the Abbey of Bury St. Edmunds, near Bedford [sic], from where it was brought to the Cambridge University Library" [1981, 53–54].

3. However, to confuse the issue the Cambridge catalog [Cambridge 1858 III, 500] claims that a short extract was published by Halliwell. In fact Halliwell published a manuscript on algorism [Halliwell 1841, 73–83], but that was the *Carmen de algorismo* (*Poem on algorism*) of Alexander de Villa Dei.

4. However, the left to right method is found in Arabic [Levey & Petruck 1965, 48–49].

5. The following symbols are employed:
   square brackets [...] enclose folio-references or indicate notes;
   angled brackets ⟨...⟩ enclose material accidentally omitted by the copyist;
   hooked brackets {...} enclose words mistakenly added by the copyist;
   round brackets (...) enclose material of an explanatory nature supplied by the translators.

6. Cf. Aristotle, *Metaphysics 987b.*

7. A bar over a Roman numeral indicates the measure is thousands, thus $\overline{\mathrm{V}} = 5000$, $\overline{\mathrm{X}} = 10{,}000$.

8. "Before" means "to the right of."

9. I.e., to the right.

10. This is the carrying procedure: whenever adding a number to one in an existing place gives more than 10, carry 1 and leave the residue in the place, e.g., 5 + 7 = 12, write 2 down and carry 1 into the next higher place (i.e., to the left).

11. If in adding you get 10, write down 0 and carry 1.

12. In this example, if adding gives 20 in the second decimal place, write down 0 and carry 2 into the third decimal place (i.e., the hundreds). Thus in the second place when computing 295 + 361 + 52, 9 + 6 + 5 = 20, we write down 0 and carry 2, giving 708 as total.

13. I.e., to the right.

14. I.e., further to the right.

15. This is the opposite direction to that currently used in the West. (It is the same as that used in earlier times in China [Lĭ & Dù 1987].)

16. Vogel [1963, 19n] claims this should be two.

17.

```
                              6 4 2 2
                              3 2 1 1
                              -------
Successive subtractions give  3
                                2
                                  1
                                    1, i.e., 3211.
```

18. The rest of this example is missing.

19. The ms. reads "accipe," probably for "incipe."

20. I.e., to the left.

21. The numbers are written down in the order indicated by the numbers in brackets alongside.

(3)	4	2	8			
(2)			2	3	2	6
(1)	2	1	4			
(6)		6	4	2		
(5)			2	3	2	6
(4)		2	1	4		
(9)			4	2	8	
(8)			2	3	2	6
(7)			2	1	4	
(12)			1	2	8	4
(11)			2	3	2	6
(10)				2	1	4

Now add (3) + (6) + (9) + (12)

4	2	8			
	6	4	2		
		4	2	8	
		1	2	8	4
4	9	7	7	6	4

22. This is the method of "casting out nines." In modern terminology, it says: if you want to check a multiplication, e.g., 125 × 42, divide both factors by 9 to get remainders 8 and 6. Multiply these remainders, getting 48 and divide by 9 getting 3 remainder. Do the original multiplication 125 × 42 = 5250. Divide by 9, getting 3 remainder which is the same. If the final remainders are not the same an error has been made.

23. The example given in the next paragraph proceeds as follows.

(6)			0			4 − 1 × 4
(5)		4				6 − 1 × 2
(4)	1					4 − 1 × 3
(3)	1					dividend
(1)	4	6	4	6	8	
(2)	3	2	4			
(12)		1	1	0		126 − 4 × 4
(11)		1	2	6		bringing up next digit
(10)		1	2			20 − 4 × 2
(9)		2				14 − 4 × 3
(8)	1	4	0			
(7)	1	4				dividend, new digit is 4
	4	6	4	6	8	
		3	2	4		
(17)			1	3	6	148 − 3 × 4 (see note [18])
(16)			1	4	[8]	20 − 3 × 2
(15)			2	[0]		11 − 3 × 3
(14)		1	1	0	8	(12) above plus last digit of dividendum
(13)	1	4	3			dividend, new digit is 3
	4	6	4	6	8	
			3	2	4	

The final remainder (17) is 136, hence the answer is $143\frac{136}{324}$.

24. The ms. actually says "lower."

25. The ms. reads "they."

26. The ms. reads "insipientes" (being unwise) instead of "incipientes" (beginning).

27. Vogel [1963, 22n] points out that the text has CXXVIII instead of the correct number CXLVIII.

28.

(4)	0	18 − 2 × 9
(3)	2	dividend
(1)	1 8 0 0	
(2)	9	
(5)	2 0	dividend, new digit is 0
	1 8 0 0	
	9	
(6)	2 0 0	dividend, new digit is 0
	1 8 0 0	
	9	

29. See the appendix to this paper (Folio 109v, 1.14).

30. Incorrect arithmetic follows since seconds ($1/60^2$) by seconds ($1/60^2$) $\times 1/60^2 = 1/60^4$, i.e., fourths, not thirds as in the text.

31. $6' \times 7' = \frac{6}{60} \times \frac{7}{60} = 42/60^2 = 42''$.
$7'' \times 9' = \frac{7}{60}^2 \times \frac{9}{60} = 63/60^3 = 63'''$
$= 1''3'''$.

32. $1\frac{1}{2} \times 1\frac{1}{2} = 90' \times 90'$
$= 8100'' = 135'$
$= 2°15' = 2\frac{1}{4}$.

33. This is an error; for "two" read "one."

34. $2°45' \times 3°10'30''$.
$2°45' = (120 + 45)' = 165'$; $3°10' = 190' = 11{,}400''$ so $3°10'30'' = 11{,}430''$.
$11{,}430'' \times 165' = 1{,}885{,}950'''$
$= 31{,}432''30''' = 523'52''30'''$
$= 8°43'52''30'''$.

35. $\frac{15}{3} \div \frac{6}{3} = 15 \div 6 = \frac{5}{2} = 2\frac{1}{2}$.

36. $10'' \div 5' = 10'' \div 300''$
$= 600''' \div 300''$
$= 2'$ (cf. note [37]).

37. This should be two minutes.

38. This should be minutes.

39. This also should be minutes.

40. This also should be minutes.

41. The correct calculation is $10'' \div 5' = 2'$, $2' \times 5' = 10''$.

42. $10' \div 5''' = 36{,}000''' \div 5'''$
$= 7{,}200°$.
$7{,}200° \times 5''' = 36{,}000''' = 600'' = 10'$.

43. Vogel [1963, 37 (n.3)] changes this to V but his errata sheet changes it back to VI.

44. $12°30'45''00'''50^{iv}$.

45. The ms. reads "invenire" (to find) instead of "minuere" (to subtract).

46. $\frac{3}{7} \times \frac{4}{9} = (3 \times 4)/(7 \times 9) = \frac{12}{63}$. The final computation which is not completed is of $3\frac{1}{2} \times 8\frac{3}{11}$.

REFERENCES

Allard, A. 1987. L'époque d'Adélard et les chiffres Arabes dans les manuscrits Latins d'arithmétique. In *Adelard of Bath: An English scientist and Arabist of the early twelfth century* (Warburg Institute surveys and texts, Vol. 14), C. Burnett, Ed., pp. 37–43. London: Warburg Institute.

Benedict, S. R. 1914. *A comparative study of the early treatises introducing into Europe the Hindu art of reckoning*. Concord, MA: Rumford.

Bolgar, R. R. 1954. *The classical heritage and its beneficiaries*. Cambridge: Cambridge Univ. Press.

Boncompagni, B. 1857. *Tratti d'arithmetica*, 2 vols. Rome: n.p.

Cambridge 1858. *A catalogue of the manuscripts preserved in the Library of the University of Cambridge*. Cambridge: Cambridge Univ. Press.

Chasles, M. 1859. [Letter.] *Comptes rendus* **43,** 1054–1061.

Curtze, M. 1898. Ueber eine Algorismus-schrift des XII. Jahrhunderts. *Abhandlungen zur Geschichte der Mathematik* **8,** 1–27.

Ershov, A., & Knuth, D. (Eds.) 1981. Algorithms in modern mathematics and computer science. *Proceedings, Urgench, Uzbek SSR, September 16–22, 1979*, Springer Lecture Notes in Computer Science, Vol. 122. Berlin: Springer.

Evans, C. R. 1977. From abacus to algorism: Theory and practice in medieval arithmetic. *British Journal for History of Science* **10,** 114–131.

Halliwell, J. O. (Ed.) 1841. *Rara mathematica*, 2nd ed. London: Parker.

Hardy, G. H., & Wright, A. M. 1938. *An introduction to the theory of numbers*. Oxford: Clarendon Press.

Haskins, C. H. 1924. *Studies in the history of mediaeval science*. Cambridge, MA: Harvard Univ. Press.

Holt, P. M., Lambton, A. K. S., & Lewis, B. (Eds.) 1970. *The Cambridge history of Islam*, Vol. 2. Cambridge: Cambridge Univ. Press.

Hughes, B., O. F. M. 1980. "De regulis generalibus": A thirteenth century English mathematical tract on problem solving. *Viator* **11,** 209–224.

——— 1986. Gerard of Cremona's translation of al-Khwārizmī's *AL-JABR*: A critical edition. *Mediaeval Studies* **48,** 211–263.

Joannes de Sacrobosco. 1521. *Algorithmus Joannis de sacro Busto*. Cracow: Haller.

Karpinski, L. C. 1915. *Robert of Chester's Latin translation of the algebra of al-Khowarizmi*. New York: Macmillan.

——— 1921. Two twelfth century algorisms. *Isis* **3,** 396–413.

Knuth, D. E. 1981. Algorithms in modern mathematics and computer science. In Ershov & Knuth [1981, 82–99].

——— 1968. *The art of computer programming*. Reading, MA: Addison–Wesley.

Lemay, R. 1977. The Hispanic origin of our present numeral forms. *Viator* **8,** 435–459.

Levey, M., & Petruck, M. (Eds. & Trans.) 1965. *Kushyar ibn Labban: Principles of Hindu reckoning*. Wisconsin University publications in mediaeval science. Madison, WI: Univ. of Wisconsin Press.

Lǐ Yǎn & Dù Shírán. 1987. *Chinese mathematics: A concise history*, J. N. Crossley & A. W.-C. Lun, Trans. Oxford: Clarendon Press.

Murray, A. 1978. *Reason and society in the Middle Ages*. Oxford: Clarendon Press.

Nagl, A. 1889. Ueber eine Algorismus-Schrift des XII. Jahrhunderts und ueber die Verbreitung der indisch-arabischen Rechenkunst und Zahlzeichen im Christl. Abendlande. *Zeitschrift für Mathematik und Physik, Historische-Litterarische Abteilung* **34,** 129–146, 161–170.

Nau, F. 1910. Notes d'astronomie Syrienne. *Journal Asiatique, 10e Sér.* **16,** 209–228.

Peet, T. E. 1923. *The Rhind mathematical papyrus*. London. Reprinted, Nendeln, Liechtenstein: Kraus, 1970.

Rosen, F. (Ed. & Trans.) 1831. *The algebra of Mohammed ben Mūsā*. London: Oriental Translation Fund.

Ross, W. D. (Ed.) 1908–1952. *Aristoteles: Works*, 12 vols. Oxford: Clarendon Press.

Saidan, A. S. 1966. The earliest extant Arabic arithmetic *Kitāb al-Fusūl fī al Hisāb al-Hindī* of Abū al-Hasan, Ahmed ibn Ibrāhīm al-Uqlīdisī. *Isis* **57**, 475–490.

——— 1978. *The arithmetic of al-Uqlīdisī*, trans. & annotated by A. S. Saidan. Dordrecht: Reidel.

Sarton, G. A. L. 1931. *Introduction to the history of science*, Vol. II (Part I). Baltimore, MD: Williams & Wilkins.

Sezgin, F. 1974. *Geschichte des Arabischen Schrifttums*, Band V. Leiden: Brill.

Smith, D. E., & Karpinski, L. C. 1911. *The Hindu–Arabic numerals*. Boston, MA: Ginn & Co.

Simpson, J. A., & Weiner, E. S. C. 1989. *The Oxford English dictionary*, 2nd ed., Vol. I. Oxford: Clarendon Press.

Steele, R. 1922. *The earliest arithmetics in English*. London: Early English Text Society (Extra Series, No. 118).

Toomer, G. J. 1973. al-Khwārizmī Abu Ja'far Muḥammad ibn Mūsā. In *Dictionary of scientific biography*, C. C. Gillispie, Ed., Vol. 8, pp. 358–365. New York: Scribner's.

Vernet, J. 1978. al-Khwārazmī. In *The encyclopaedia of Islam*, C. E. Bosworth, E. van Donzel, B. Lewis, & Ch. Pellat, Eds., new ed., Vol. 4, pp. 1070–1071. Leiden: Brill.

Vogel, K. 1963. *Mohammed ibn Musa Alchwarizmi's Algorismus, Das fruehste Lehrbuch zum Rechnen mit indischen Ziffern*. Aalen Osnabrück: Zeller.

Yushkevich, A. P. (Juschkewitsch) 1964a. *Geschichte der Mathematik im Mittelalter*. Leipzig: Teubner.

Yushkevich (Youskevitch), A.-P. 1964b. Über ein Werk des Abū Abdallāh Muḥammad ibn Mūsā . . . zur Arithmetik der Inder. In *Beiheft zur Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin, Beiträge zur Geschichte der Naturwissenschaften, Technik und Medizin*, herausgegeben zum 60. Geburtstag Gerhard Harigs von I. Strube und H. Wussing, pp. 21–63. Leipzig: Teubner.

Zemanek, H. 1981. DIXIT ALGORIZMI: His background, his personality, his work, and his influence. In Ershov and Knuth [1981, 1–81].