The Base

Figure out what is happening here. All of these statements are true.

$17_{10} = 21_{8}$	$17_{10} = 32_{5}$	$17_{10} = 101_{4}$	$17_{10} = 10001_{2}$
$17_{10} = 13_{14}$	$17_{10} = 12_{15}$	$17_{10} = 11_{16}$	$17_{10} = 10_{17}$

Bases refer to the base in an exponential equation. The base of a number system is notated with the subscript seen here. 21_8 is 21 Base-8.
And 17_{10} is just our regular 17 in Base-10, our normal decimal system.

$$21_8 = 17_{10}$$
$$2 \times 8^1 + 1 \times 8^0 = 1 \times 10^1 + 7 \times 10^0$$
$$16 + 1 = 10 + 7$$

Our normal base system is base-10 and uses these symbols:

0	1	2	3	4	5	6	7	8	9

This is slightly annoying to me because base-10 uses 0-9, not 1-10. Like the wheat-and-the-chessboard, we are once again cursed by counting things starting at 0. If you recall, the 64th square in that problem had an exponent of 63 because we started with 0.

In our standard decimal, base-10, system. The 10th digit is 9.

Technically speaking, any base over base-10 requires additional number symbols.
For example: The standard way to notate in base-16 utilizes these symbols:

B-16	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
B-10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Converting in and out of B-10 is not difficult, but it can be laborious.

First of all, let's review our own decimal system.

$$2609.3 = (2 \times 1000) + (6 \times 100) + (0 \times 10) + (9 \times 1) + \left(9 \times \frac{1}{10}\right)$$

Placement is everything.

10_3	10_2	10_1	10_0	10_{-1}
2	6	0	9	3

That's why the zero (*cifre or sifr*) is so historically important. It's not that people in the ancient world didn't under the concept of nothing, it's that they typically didn't employ an exponential place system like our modern system.
Without a zero you can't easily communicate the difference between 2609.3 and 26 9.3 and 269.3.

And in Roman numerals these large numbers are unweildy and inconsistant in length.
MMDCIX = 2609 vs. MCMLXXXVIII = 1988
Imagine multiplying these two numbers together using only Roman Numeral notation.

For the sake of demonstration, let's work a little with Base-4, B-4.

Counting to 15_{10} in B-4

B-4	0	1	2	3	10	11	12	13	20	21	22	23	30	31	32	33
B-10	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Converting from B-4 to B-10

This is pretty easy. Just set up a table like so and input your B-4 number in the proper column.

For example: What is 2031_4 in B-10. [The subscripts indicate base system]

$4^3 = 64$	$4^2 = 16$	$4^1 = 4$	$4^0 = 1$
2	0	3	1
$2 \cdot 64 = 128$	$0 \cdot 16 = 0$	$3 \cdot 4 = 12$	$1 \cdot 1 = 1$

In B-10 this is: 128 + 0 + 12 + 1 = 141

So,

$2031_4 = 141_{10}$

Converting the other direction is more annoying.... from B-10 to B-4.

E.g. What is 282_{10} in B-4?

Start again with a table and make sure you have powers of 4, which are just under the B-10 number you want to convert. Then just split up your B-10 number and fit it into this table by doing division with remainders, which I notate R?. E.g. R3 means remainder of 3. Start with the left-most column and work to the right.

$4^4 = 256$	$4^3 = 64$	$4^2 = 16$	$4^1 = 4$	$4^0 = 1$
282/256 = **1** with R26	26/64 = **0** with R26	26/16 = **1** with R10	10/4 = **2** with R2	2/1 = **2** with R0
1	0	1	2	2

$282_{10} = 10122_4$

Now notice a couple of things from these two conversions:

$141_{10} = 2031_4$

$282_{10} = 10122_4$

The two B-10 numbers are in the ratio 1:2. Not so with the B-4 numbers.... or are they?

Let's multiply 2031_4 by 2... by adding it to itself:

Remember there is no such thing as a 6 or a 4 in B-4.

[$6_{10} = 12_4$ and $4_{10} = 10_4$]

$$\begin{array}{r} 2031_4 \\ +\ \underline{2031_4} \\ 10122 \end{array}$$

I did this in the video with a few more bells and whistles:
See Video: "Base Systems Vid"

Exercise 1

Base-4 Multiplication Table

Fill in all empty squares with the appropriate Base-4 numbers.
It's weird at first, but you'll get the hang of it with practice.
Shown in Video: "Base Systems Vid"

B-10		0	1	2	3	4	5	6	7	8
	B-4	0	1	2	3	10	11	12	13	20
0	0		0						0	0
1	1	0	1	2	3	10	11	12		20
2	2				12			30	32	100
3	3	0	3				33		111	120
4	10			20	30		110	120	130	200
5	11						121			
6	12						132	210		300
7	13						203	222		320
8	20		20	100		200				1000

Are prime numbers in base-10 still prime numbers in base-4?

Definition: A Prime Number is a natural number (counting number or positive integer) **greater than 1 that cannot be formed by multiplying two smaller natural numbers.** Put another way, a prime number is a natural number that has exactly two distinct natural number factors (or divisors), namely 1 and the number itself.

Examine these two tables of numbers.
They are identical, except that one is in base-10 and the other in base-4.
Are prime numbers in base-10 (shaded) still prime in base-4?
You may find the multiplication table for base-4 useful in determining some of this.

Base-10 Primes

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23

Base-4 Primes?

0	1	2	3
10	11	12	13
20	21	22	23
30	31	32	33
100	101	102	103
110	111	112	113

Prime factorize each of the numbers.
E.g. In Base-10 the prime factorization of 14 is 2 x 7.
$14_{10} = 32_4$ $[3 \cdot 4^1 + 2 \cdot 4^0]$

If you look at the Base-4 Multiplication Table on the previous page,
you'll see that 32_4 is made by multiplying
$13_4 \cdot 2_4$

You may note that 13_4 and 2_4 are prime numbers in the Base-4 system. You see this because they only appear from multiplications involving 1 and themselves.
That's basically the definition of a Prime Number. [See above.]
All the other shaded numbers behave similarly.
So yes, a Prime Number in Base-10 is a Prime Number in Base-4.

Also note that all of the Prime Numbers written in Base-4 appear in this list of Prime Numbers written in Base-10 even though technically they don't refer to the same numbers.

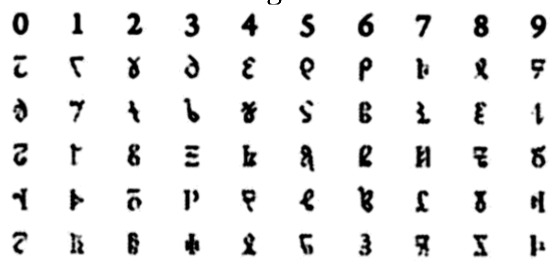
The First Primes Up to 113 (in Base-10)

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
53	59	61	67	71	73	79	83	89	97	101	103	107	109	113

Now for the last part,

Sexagesimal... Base-60.

Here are the digits used: 0 to 60



We use B-60 all the time..... for telling TIME.

When I write 2:45:15
...you know that it means, 2 hours, 45 minutes, and 15 seconds,
where hours are 60_2 seconds, minutes are 60_1 seconds, and seconds are 60_0 seconds.
Because nobody I know has memorized all 60 characters (shown above) of the sexagesimal system we do a hybrid thing. We use the decimal numbers from 0-59 and separate each exponential column with a ":".
It would be like us writing the number 318 like this: 3:1:8.

The sexagesimal system works like this:

	hours	minutes	seconds
	$60_2 = 3600$	$60_1 = 60$	$60_0 = 1$
Input B-60 Number here:	2	45	15
B-10 Equivalent in Seconds	$2 \cdot 60^2$	$45 \cdot 60^1$	$15 \cdot 60^0$
B-10 Equivalent in Seconds	7200	2700	15
B-10 Equivalent in Seconds all Summed Up	$0 + 7200 + 2700 + 15 = 9915\ seconds$		

2:45:15 = 2 hours, 45 minutes, and 15 seconds.

This equals 9915 seconds = 9 thousand, 9 hundred, 1 ten, and 5 ones.

In our digital system we have named the columns: ones, tens, hundreds, thousands, etc.
In the sexagesimal the columns also have names: seconds, minutes, hours.

Trivia: The word "minute" refers to a *min-oot* part... a *mine-oot* part of an hour.

Pronounced: mī′n(y)o͞ot

The word "second" is the second *mine-oot* part, with the minute being the first *mine-oot* part.

Now, to be a bit more complete,
...we actually use a horribly mixed-up system for time once you get beyond hours.

	weeks	days	hours	minutes	seconds
	$7 \cdot 24 \cdot 60^2 =$ $= 604{,}800$	$24 \cdot 60^2 =$ $= 86{,}400$	$60_2 = 3600$	$60_1 = 60$	$60_0 = 1$
Input weeks-days-hrs:min:sec	5	3	2	45	15
B-10 Equivalent in Seconds	$5 \cdot 7 \cdot 24 \cdot 60^2$	$3 \cdot 24 \cdot 60^2$	$2 \cdot 60^2$	$45 \cdot 60^1$	$15 \cdot 60^0$
B-10 Equivalent in Seconds	3,024,000	259,200	7200	2700	15

5weeks:3days:2:45:15

5 weeks, 3 days, 2 hours, 45 minutes, and 15 seconds is **3,293,115** seconds in Base-10.

Similarly, this <u>mixed</u> system applies to how we measure angles.
Minutes and seconds are Base-60, but degrees are 360.

	Degrees	minutes	seconds
	$360 \cdot 60^1$	$60_1 = 60$	$60_0 = 1$
Input degrees:min:sec	90º	45'	15"
B-10 Equivalent in Seconds	$90 \cdot 360 \cdot 60^1$	$45 \cdot 60^1$	$15 \cdot 60^0$
B-10 Equivalent in Seconds	1,944,000	2700	15

This adds up to 1,946,715 seconds. However, this doesn't really help us much in terms of getting a feel for the measurement of an angle. I have no idea how big an angle is if it has 1.9 million seconds. So, instead of converting to seconds, let's convert to degrees, but in decimal notation, rather than all these minutes and seconds.

We can do this intuitively.
90º 45' 15" is not hard to imagine in decimal because of our familiarity with time.
45' is just 45/60 of a degree, and 15" is just 15 sixtieths of a sixtieth of a degree, $15/60_2$ of a degree.
So that should end up being...

$$90^\circ + (45/60)^\circ + (15/60_2)^\circ =$$

$$90° + 0.75° + 0.0041\overline{66} = 90.7541\overline{66}°$$

This makes sense. 90º 45' 15" is not quite 91º. It's $90.7541\overline{66}°$.

So here is how we callibrate our conversions to degrees rather than seconds.
Make degrees the 1.... the ones column... and the minutes and seconds fractions.

	Degrees ones	**minutes** sixtieths	**seconds** three thousand six hundreths
	$1 = 360^{\emptyset}$	$\frac{1}{60} = 60^{-1}$	$\frac{1}{60^2} = 60^{-2}$
Input degrees:min:sec	90º	45'	15''
B-10 Equivalent in degrees	$90 \cdot 1$	$45 \cdot 60^{-1}$	$15 \cdot 60^{-2}$
B-10 Equivalent in degrees	90	0.75	$0.0041\overline{66}$

Add it all up to get $90.7541\overline{66}$°. Just like we figured.

Your homework for Wednesday,
is to make the following spreadsheet converter
from any Base to Base-10.
To submit it as HomeWork just convert this Base-13 number to Base-10 and send me the Base-10 result:

What is 0:11:12:5$_{13}$ in Base-10?

Hint: Now.

There is a video tour is this spreadsheet posted on our site.

An Excel Spreadsheet Base-System Converter
Base-x to Base-10

Here is what the converter looks like superficially. You input a base in cell B3. Then input a number in that base system in cells D5, E5, F5, and G5.

In this example I chose a base-3 system and the number 2102_3 in this system.

The output in base-10 is 65. Thus, 2102_3 in base-3 is 65_{10} in base-10.

$$2102_3 = 65_{10}$$

	A	B	C	D	E	F	G	H
1	**Base-x to Base-10 converter**							
2								
3	Input Base-x:	3		x^3	x^2	x^1	x^0	
4				27	9	3	1	
5			Input Base-x Number:	2	1	0	2	
6		0 through	2					
7								
8			Output Base-10:	54	9	0	2	
9			Base-10 Total:	65				
10								

Below is what each cell actually contains. This is what you should duplicate in your Excel or GoogleSheet spreadsheet.

This is not what you normally see, this is what has been typed into each cell.

	A	B	C	D	E	F	G
1	**Base-x to Base-10 converter**						
2							
3	Input Base-x:	3		x^3	x^2	x^1	x^0
4				=B3^3	=B3^2	=B3^1	=B3^0
5			Input Base-x Number:	2	1	0	2
6		0 through	=B3-1				
7							
8			Output Base-10:	=D5*D4	=E5*E4	=F5*F4	=G5*G4
9			Base-10 Total:	=SUM(D8:G8)			
10							

When you type an "=" in a cell Excel immediately wants a formula, like =SUM(D8:G8). After inputting a formula you must hit "Return/Enter" in order to get out of formula-editing mode. If you don't hit "Return/Enter" Excel will think that you are still entering stuff into the formula. You'll figure it out... but it will annoy you at first. It takes time for Excel to train you. If you feel caught inside Excel hell, usually you can just hit "Return/Enter" and it will end.

Here are explanations of all the formulas in this spreadsheet...

=B3^3 This formula takes the contents of cell B3 and sets it to the 3rd power.
=D5*D4 Multiplies the contents of D5 by the contents of D4.
=SUM(D8:G8) Finds the sum of D8, E8, F8, and G8.
=B3-1 Takes the number in B3 and subtracts 1 from it.

A1 just has text in it. Just type in the text.
Cells D3-G3 also just have text in them as far as Excel is concerned. Just type it in.

Same with the other cells like A3, B6, C5, C8, C9

Your input cells are B3, and D5-G5. In B3 you input whatever base you want to work with. In D5-G5 you input the number in the base system you have chosen. Remember, if you choose Base-6, you can only input 0-5. If you choose Base-9, you can only input 0-8... if you choose Base-10, you can only input 0-9, etc. So be careful when inputting your number to make sure that it is a number that makes sense in the base-system you have chosen. It doesn't make sense to input 324 if you are working in Base-2. You would need to input a number with only 0s and 1s, like 1 0 1 1.

I also changed the color of all of the input cells, the cells where you type in input numbers: B3 and D5-G5. I did this so that you are reminded that you only input numbers into the shaded cells and that you should never type inputs into the unshaded cells... which could potentially ruin the functionality of your spreadsheet.

Optional: Once you get the sheet working properly can select the whole thing (A1-G9), copy it, and paste another copy of it right next to your original. This way you could compare one system to another. If you are really sophisticated, you could link them together somehow. [When pasting, functionality can sometimes be damaged. You might have to fiddle with stuff to make the new copy work.]

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Base-x to Base-10 converter								Base-x to Base-10 converter						
2															
3	Input Base-x:	-10		x^3	x^2	x^1	x^0		Input Base-x:	-100		x^3	x^2	x^1	x^0
4				-1000	100	-10	1					-1000000	10000	-100	1
5			Input Base-x Number:	3	3	3	3				Input Base-x Number:	3	3	3	3
6		0 through	-11							0 through	-101				
7															
8			Output Base-10:	-3000	300	-30	3				Output Base-10:	-3000000	30000	-300	3
9			Base-10 Total:	-2727							Base-10 Total:	-2970297			
10															
11	Base-x to Base-10 converter								Base-x to Base-10 converter						
12															
13	Input Base-x:	10		x^3	x^2	x^1	x^0		Input Base-x:	100		x^3	x^2	x^1	x^0
14				1000	100	10	1					1000000	10000	100	1
15			Input Base-x Number:	3	3	3	3				Input Base-x Number:	3	3	3	3
16		0 through	9							0 through	99				
17															
18			Output Base-10:	3000	300	30	3				Output Base-10:	3000000	30000	300	3
19			Base-10 Total:	3333							Base-10 Total:	3030303			

Once you are comfortable with this converter, feel free to expand it to go up to x^10 and down to x^-10 or some other big spread. There are lots of short cuts you will learn and/or discover by playing with this basic converter.