Reading and Exercises: How to use hot water as a clock: Newton's Law of Cooling

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The general formula for cooling: $T(t) = Ce^{-kt} + T_{amb.}$

Where...

- $T(t)$ is Temperature at time, t.
- T_{amb} is the ambient temperature (room temperature)
- $T(0)$ is the initial temperature of the water in the container.
- t is elapsed time starting at 0.
- e is Euler's number (ca. 2.718) [more precisely given on a calculator]
- k is the cooling rate [like the interest rate]
- C is the initial temperature difference: $T(0)$ - T_{amb}

For the situation I'd like you to model, the $T_{amb} = 19^oC$

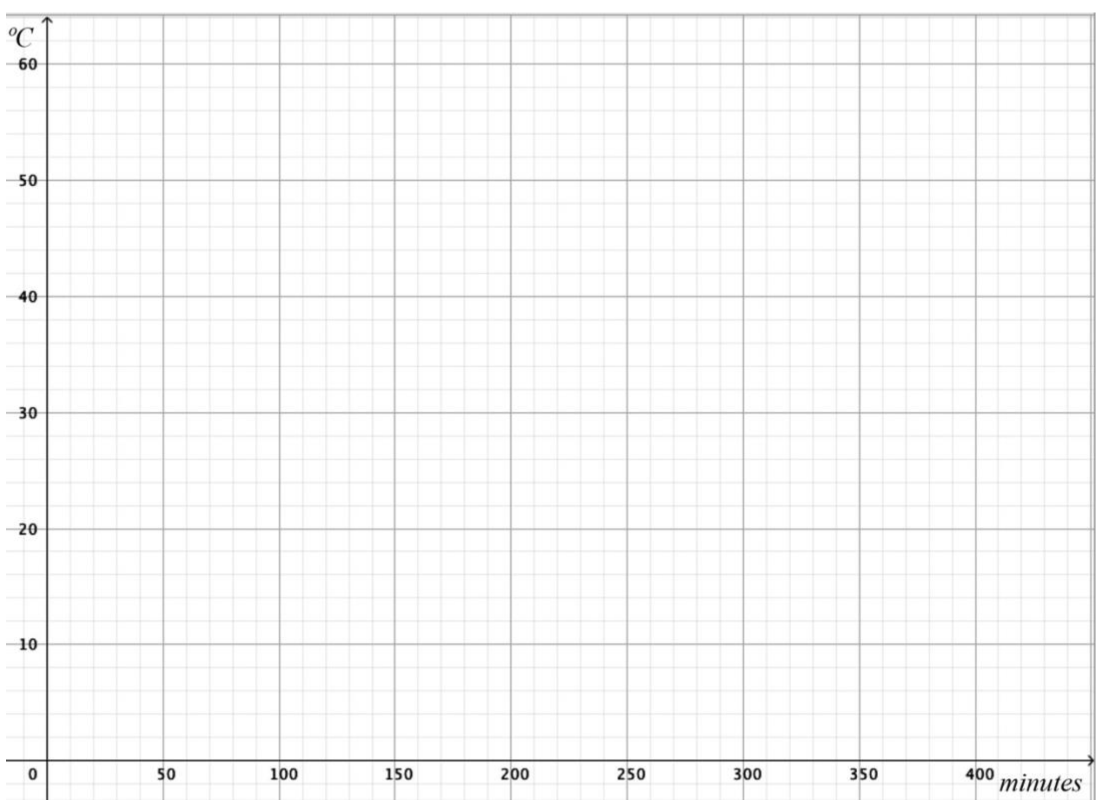
[That was the temperature of the room when I made the video.]

1) $T_{amb}=19^oC$.

2) Figure out your $T(0)$, the temperature at $t = 0$.
To do this, watch the video, determine the starting temperature and time you want to use as your starting point... your initial conditions.
You could choose the temperature at 4:00 or 4:15, or 4:25, or 5:00. It's your choice, but whatever you choose, call that t = 0 and calibrate all subsequent times to that starting point.
E.g. If you choose to start at 4:35, call that t = 0. The temperature at 4:35 appears to be 42º. You now have a point, (0, 42º). Then at 5:00. t will equal 25 minutes... at 5:30, t = 55 minutes, at 6:35, (2 hours later) t = 120 minutes.... etc.

3) Fill in the table below with at least 15 data points and then make a graph using these points. Record data all the way to the end of the video.

Clock Time	Elapsed minutes, t	Temp. ºC
	0	



$$T(t) = Ce^{-kt} + T_{amb.}$$

4) Now figure out C, which is the starting point temperature minus the ambient temperature. T(0) – $T_{amb.}$

I gave you the ambient temperature, *T_{amb}=19ºC.* Now use the temperature you chose as your starting temperature in step 2, written in the first row of the table above. Depending on what you chose for your starting temperature your value for C will probably range between 27º and 37º.

Mathematical observation: You are basically just solving for C when t = 0.

$$T(0) = Ce^{-k\cdot 0} + T_{amb.}$$

This is possible when t = 0 because the exponent becomes 0, and thus $e^0 = 1$, leaving you with...

$$T(0) = C + T_{amb.}$$

That's why we want a well-defined t = 0. It allows us to find C.

You were given $T_{amb.}$ ... it's 19º. So...

$$\boldsymbol{C = T(0) - 19°}$$

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5) Now pick three times with corresponding Temperatures from your data set from the table above. Write them in the form, (t, T). E.g. (18min, 40º). Pick a time/Temp. point somewhere in the approx. 20 minute area, another from the approx. 120 minute area and another from the approx. 240 minute area.

$t = 20min \pm 5min$	$t = 120min \pm 15min$	$t = 240min \pm 30min$
time, Temp. #1	time, Temp. #2	time, Temp. #3

You will use these three point to come up with three different cooling rates, *k*.

Here is an example of how you find *k*.
The following example numbers do not correspond to our water-cooling video.
Hypothetical point for calibrating *k*: (100', 25º), hypothetical T_{amb} = 10º, and hypothetical C = 20º.

$T(100) = 25° = 20°e^{-k\cdot 100} + 10°$	Now solve for k.
$e^{-k\cdot 100} = \frac{25° - 10°}{20°}$	First rearrange with some algebra.
$\ln(e^{-k\cdot 100}) = \ln\left(\frac{15°}{20°}\right)$	Take the natural log, *ln*, of both sides.... a little calculator work on the right side...
$-100k \approx -0.287682$ $\boldsymbol{k = 0.00287682}$	Here's the trick. $\log_b b^x = x$ Example: $\log_{10} 1000 = \log_{10} 10^3 = 3$
Ta Da. Now you have a value for the cooling rate, calibrated to the point (100', 25º). It's good to get these values pretty accurately. I recommend at least 6 decimal places.	

In general the formula for *k* is:

$$k = \frac{\ln\left(\frac{T(t) - T_{amb}}{C}\right)}{-t}$$

[This might come in handy when making a spread sheet out of this.]

6) Now make a complete set of tables for all the things you calculated.

time, Temp. #1	time, Temp. #2	time, Temp. #3
k_1 =	k_2 =	k_3 =

Put your other results (from above) here.

T_{amb} = 19º [given]
T(0) =
C =

$$T(t) = Ce^{-kt} + T_{amb}.$$

7) Finally, put it all together to make three equations, one for each value of k.

Equation 1	Equation 2	Equation 3
$T(t) =$	$T(t) =$	$T(t) =$

Here's an example for making the final *T(t)* equation, drawn from the previous example.

Hypothetical point for calibrating *k*: (100', 25º), hypothetical T_{amb} = 10º, and hypothetical C = 20º.

[and this implies that T(0) = 30º] Resultant *k*: $k = 0.00287682$

Resultant Equation: $\boldsymbol{T(t) = 20°e^{-(0.00287682)t} + 10°}$

Test Drive 1: $T(60min) = 20°e^{-(0.00287682)60} + 10° = 20°e^{-0.1726074} + 10° \approx 26.8°$

Test Drive 2: $T(0) = 20°e^{0} + 10° = 30° \quad \ldots as\ expected.$

8) Now graph them up on the graph on p. 1 and see how well they match the original data taken from the video. Notice where the curves intersect.

9) Now....Choose the best fit equation: Equation 1, 2, or 3? You be the judge.

Using this equation, solve for time, *t*. This will give you an equation in the form *t(T)*, allowing you to input a Temperature and get time for an output.

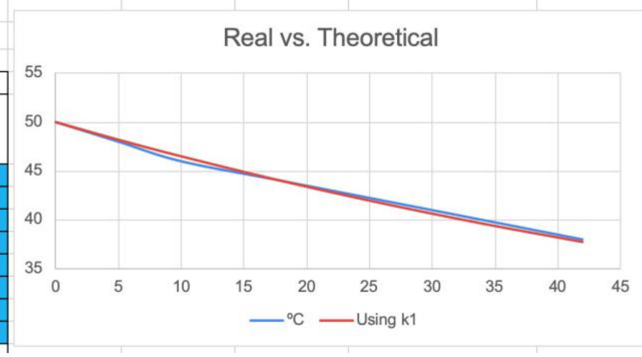
$$t(T) =$$

This form of the equation might be useful for determining "time of death" or even the age of something, as in Carbon-14 dating.

10) Now, to wrap this all up... make a spreadsheet for all of this. I made a tutorial video on how to do it here: http://www.mifami.org/Math110/NewtonianCoolingSpreadsheetConstuct.mp4

Here's an example of how it should be set up. Feel free to change this structure if you are inspired. Send me a screen shot when you have it completed and working. Keep in mind, all the inputs shown here in blue might be different from what you use.

	A	B	C	D	E	F	G	H	I	J	K
1	**Newton's Cooling: Real vs. Theoretical**							time, t	Temp, T		
2	T_{amb} =	19	= room Temp. in °C				k_1 data point	18	44	0.011950632	= k_1
3	T(0) =	50	= Temp. when I started the model in °C				k_2 data point	etc.	etc.	etc.	= k_2
4	C =	31	= difference between initial Temp. and room Temp. in °C				k_3 data point	etc.	etc.	etc.	= k_3
5											
6											
7				$T(t) = Ce^{-kt} + T_{amb}$							
8											
9	Data from Video			Theoretical Models for T(t)							
10	Clock Time	Elapsed time, t, minutes	°C	Using k1	Using k2	Using k3					
11	4:10	0	50	50	etc.	etc.					
12	4:15	5	48	48	etc.	etc.					
13	4:20	10	46	47	etc.	etc.					
14	4:28	18	44	44	etc.	etc.					
15	4:36	26	42	42	etc.	etc.					
16	4:44	34	40	40	etc.	etc.					
17	4:52	42	38	38	etc.	etc.					
18	etc.	etc.	etc.		etc.	etc.					



Exponents: Need to know properties.

[for x, y, a *and* b $\mathbb{R}$*eal Numbers* (additional limitations apply)]

$a^x a^y = a^{x+y}$	$(ab)^x = a^x b^x$	$\sqrt[x]{a} = a^{1/x}$
$\frac{a^x}{a^y} = a^{x-y}$	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	$\frac{1}{a^x} = a^{-x}$
$(a^x)^y = a^{xy}$	$a^0 = 1$	$\frac{1}{a^{-x}} = a^x$

A LIST OF THE PRIMARY LOG PROPERTIES.

1) $x = b^y$ is the same as $y = \log_b x,$

2a) $\log_b(xy) = \log_b x + \log_b y$

2b) $\log_b \frac{x}{y} = \log_b x - \log_b y$

3) $\log_B A = \frac{\log_c A}{\log_c B}$

4) $c \log_b x^a = \log_b x^{ac}$

...and...

$\log_b b^y = y$ and $b^{\log_b x} = x$

Eq.3) $\log_B A = \frac{\log_c A}{\log_c B}$

11) Using equation 3, prove that $\log_A B = \frac{1}{n}$ can be rewritten $\log_B A = n$

Solution: $\log_A B = \frac{1}{n} = \frac{\log_B B}{\log_B A}$ $\rightarrow$ *cross multiply* $\rightarrow$ $\log_B A = n \log_B B = n$

12) Exercise: Just do some math. Solve for the variable.

E.g. $\ln(3x) = 10$ $\ln(3) + \ln(x) = 10$ $\ln(x) = 10 - \ln(3) \cong 8.901$ $e^{8.901} = x \cong 7342.16$	a) $\ln(16x) = 3$
b) $\log_4 x + \log_4(x - 6) = 2$ Hint: Use log product property (Eq. 2a) and then some basic factoring.	c) $e^{2x} - 2e^x - 8 = 0$
d) $5^x = 12.328$	e) $4e^{2x} = 20e$
f) $e^{4x} - 7e^{2x} + 12 = 0$	Take a break.

Answers to above:
a) about 1.255
b) 8, -2
c) $x \cong 1.39$ and the other solution is not in domain.
d) about 1.56
e) about 1.30
f) $x \cong 0.55$ *and* 0.69

Radio Carbon Dating in a Nutshell

--Carbon in the atmosphere is absorbed by plants. This is part of photosynthesis.
--Cosmic rays cause standard ^{12}C to change into ^{14}C. Basically, a standard ^{12}C picks up two neutrons, making it sort of a heavy version of carbon. It is somewhat unstable in this configuration and when it is not being bombarded continually by cosmic rays, it tends to relax into a more stable state which is actually ^{14}N (nitrogen). Roughly speaking, ^{14}C (6 protons, 8 neutrons, and 6 electrons) goes through beta decay (neutron splits into an electron and a proton) to ^{14}N (7 protons, 7 neutrons, and 7 electrons).
--Most of the carbon in a living plant or animal is ^{12}C. However, about 1 in a million carbon atoms is the heavier ^{14}C, this cosmically souped up carbon. However, this heavy carbon is somewhat unstable and decays away at a statistically constant rate.
--The half-life of ^{14}C is about 5700 years. This means that in 5700 years half of the ^{14}C will transform into something else. In this case it will transform into ^{14}N (nitrogen), but the details of this are not necessary here. [^{14}N (nitrogen) has 7 electrons, 7 protons, 7 neutrons]
--This means that after the plant or animal has been dead for about 5700 years its ^{14}C to ^{12}C ratio will be 1 in 2 million rather than 1 in 1 million... half the level that it was when it was alive. Its "half-life."

Carbon 14 dating is effective up to about 50,000 years. The ratio at this point is about 1 in 100 million. At this point there isn't enough ^{14}C to measure accurately.

Radio Carbon Dating: The half-life of ^{14}C is about 5730 years. The ratio of ^{14}C :^{12}C in a fresh plant is approximately 1:1,000,000. [One atom of ^{14}C per 1 million atoms of ^{12}C.] You are given a sample of wood from a beam from an archeological site. The measurement of its carbon isotopes yields a ratio of ^{14}C :^{12}C is 0.5:1,000,000. [One half parts per million.] In other words, its ^{14}C content is one half of what a freshly cut piece of wood would have.

So let's put all this data into our exponential equation and solve for the decay rate, r. The initial amount is 1 (one part per million). The resultant amount, A(t), is 1/2.... 0.5 per million (... or 1 per 2 million). And the time is 5730 years. [Because that's the experimentally determined of half-life of ^{14}C.]	$A(t) = P_0 e^{rt}$ $1/2 = 1e^{5730r}$
Solve for the rate of decay, r, by rewriting into log form.... $x = b^y$ is $y = \log_b x$. Then just solve for r.	$1/2 = e^{5730r}$ $\ln 1/2 = 5730r$
That's the rate of decay. It's in the exponent of the compound interest formula. It's negative because it diminishes quantities instead of growing them. That negative sign puts e^{rt} in the denominator. It's now a fraction.	$r \cong -0.000121$
Here is the radioactive decay formula for ^{14}C: **In most cases P_0 = 1, because that is the initial state of "1" in a million.**	$A(t) = P_0 e^{-0.000121t}$ $\boldsymbol{A(t) = e^{-0.000121t}}$
In order to determine the age of a sample, you need to solve for t.	
13) Solve the radioactive decay formula of ^{14}C for t.	

You might be slightly confused about these ratios. In the decay formula we have A(t) and we have P_0. P_0 is the initial Carbon-14 amount as compared with the amount of Carbon-12.
This ratio is 1 per 1,000,000.
A(t) is the resultant amount of Carbon-14 after some amount of time, t. It is also the ratio of Carbon-14 to Carbon-12. For example, it might be 0.5 per 1,000,000
Because A(t) is on one side of the equal sign and P_0 is on the other, it looks like this: (using the example)

$$\frac{A(t)}{1{,}000{,}000} = \frac{0.5}{1{,}000{,}000} = \frac{1}{1{,}000{,}000}e^{-0.000121t}$$

The $\frac{1}{1{,}000{,}000}$ cancel out and we are left with...

$$A(t) = 0.5 = 1e^{-0.000121t}$$

14) Given the ratios of ${}^{14}C:{}^{12}C$, figure out how old the samples are.

[All of the following problems assume that P_0 is 1.]

a) You are given a desiccated tissue sample from Siberia which yields a ratio of $0.12\,{}^{14}C$ per 1million ${}^{12}C$.   Hint: A(t) is 0.12. The answer's first two digits are 1 and 7.
b) You are given fragments of a papyrus scroll found in Egypt. The measured ratio is 0.7:1million.   Hint: third digit is 4.
c) You are given some bone fragments from an unknown animal. The measured ratio is 0.002:1million. [This ratio is about as small as can be accurately measured. Carbon 14 dating is unreliable much beyond this.]   Hint: first digit is 5 and the third is 3.

Here is a website you can use to check your answers: https://www.math.upenn.edu/~deturck/m170/c14/carbdate.html

Editor's Note:
You might be slightly confused about these ratios. In the decay formula we have A(t) and we have P_0.
P_0 is the initial Carbon-14 amount as compared with the amount of Carbon-12. This ratio is 1 per 1,000,000.
A(t) is the resultant amount of Carbon-14 after some amount of time, t. It is the ratio of Carbon-14 to Carbon-12 at the designated time. For example, it might be 0.3 per 1,000,000.

Because A(t) is on one side of the equal sign and P_0 is on the other, it looks like this: (using the example)

$$A(t) = \frac{0.3}{1{,}000{,}000} = \frac{1}{1{,}000{,}000}e^{-0.000121t}$$

The $\frac{1}{1{,}000{,}000}$ cancels out and we are left with...

$$A(t) = 0.3 = 1e^{-0.000121t}$$

After about 55,000 years, Radio Carbon Dating is not useful. Then you move on to Uranium–thorium dating which can go back about 500,000 years. Then to Uranium–lead dating which has a range of about 1 million to 4.5 billion years. Or, if you really want to go retro, use Rubidium–strontium dating which can take you back more than 50 billion years.... well beyond the Big Bang. The general concepts are all the same– radioactive decay and an exponential function. What changes is the rate of decay, the r. Stuff decays off at a constant rate (it is assumed). The formula for cooling water and radioactive decay are both essentially the same. They differ in their rates of change. With radioactive decay you are just measuring how long it takes to metaphorically cool off.

A few other methods for dating:

-Uranium-thorium dating [using the decay rate from uranium-234 to throium-230] Uranium-thorium dating has an upper age limit of somewhat over 500,000 years, defined by the half-life of thorium-230.

-Uranium–lead (U–Pb) dating is one of the oldest and most refined radiometric dating method, with a routine age range of about 1 million years to over 4.5 billion years

-Rubidium-strontium dating is a radiometric dating technique used to determine the age of rocks and minerals from isotopes of rubidium ($_{87}$Rb) and strontium ($_{87}$Sr, $_{86}$Sr). $_{87}$Rb (one of two naturally occurring isotopes of rubidium) decays to $_{87}$Sr with a half-life of 48.8 billion years, well beyond the current estimate for the age of the universe.

Other methods...

-Dendrochronology- tree rings. Often used to corroborate Carbon-14 dates.
-Vendange- vineyard records go back thousands of years in some places.
-Sediment deposits in rivers and lakes leave measurable time-lines.
-Glacial evidence

Overview of exponential equations we've been using in the past couple of classes.

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ Where *A* is the final amount, *P* is the principal, *r* is the interest rate per time period, *n* is the number of times the interest rate is compounded per time period, and *t* is the amount of time in total.	$A(t) = Pe^{rt}$ As if *n*, in the equation to the left, went to infinity. Continuously compounded interest.	$A(t) = P_0e^{-0.000121t}$ Radioactive decay formula for ^{14}C, Where A is the amount of ^{14}C compared to ^{12}C. P_0 is usually 1 in these problems.

Optional

Derivation of Newton's Law of Cooling

$\frac{dT}{dt} = -k(T - T_{amb})$	Observation: Temperature changes over time less and less as the difference between the Temp. of the object and the Ambient Temp. decreases.
$\frac{1}{(T - T_{amb})} dT = -kdt$	Algebra
$\int \frac{1}{(T - T_{amb})} dT = \int -kdt$	Calculus
$\ln\|T - T_{amb}\| + D = -kt + E$	Calculus $\int \frac{1}{x} dx = ln\|x\| + C$
$\ln\|T - T_{amb}\| = -kt + F$	Combine constants (E and D) into new constant, F, just to clean things up.
$e^{\ln\|T - T_{amb}\|} = e^{-kt+F}$	Algebra
$\|T - T_{amb}\| = e^{-kt}e^{F}$	Algebra. Pick off another constant ...call e^F, C.
$\|T - T_{amb}\| = Ce^{-kt}$	If $T > T_{amb}$ then absolute value is not needed. This is the cooling off scenario. [$T < T_{amb}$ is the heating up scenario]
$T(t) = Ce^{-kt} + T_{amb}$	Newton's Law of Cooling