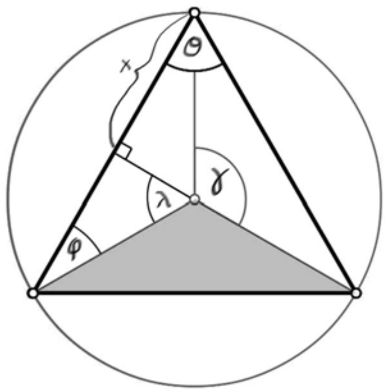
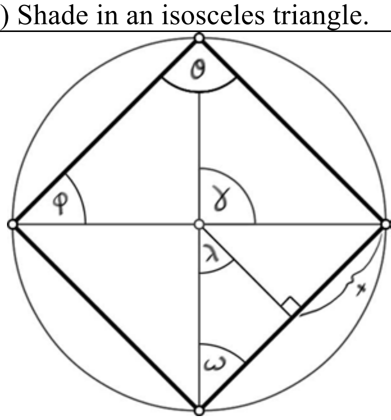
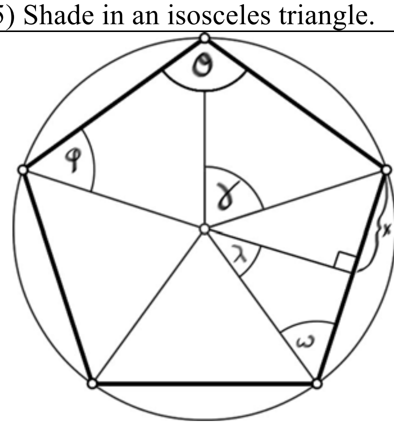
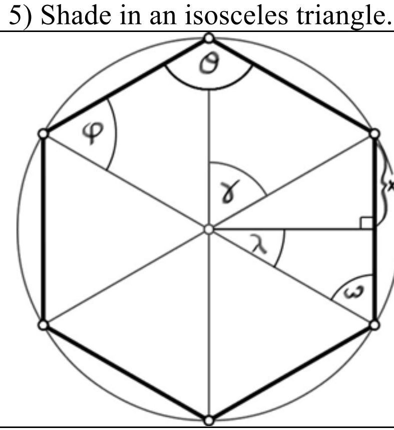
BLC190-ATF [mifami.org] CLASS 7 BASIC GEO-TRIGONO-METRY



SOME MATHEMATICS PERTAINING TO REGULAR POLYGONS INSCRIBED IN A CIRCLE

A Few Mathematical Properties of an Equilateral Triangle Inscribed in a Circle	Example: equilateral triangle (see above)
1) All interior angles of a **triangle** add up to 180º.	$\theta = 60º$
2) Interior angles (θ) are bisected by radii drawn from the center of the circle to a vertex.	$\varphi = \frac{\theta}{2} = \frac{180º-\gamma}{2} = 30º$
3) Angles between rays (γ) which extend to vertices measure 360º/3.	$\gamma = 120º$
4) Angles formed by drawing a radius to the middle point of any side of the triangle will be perpendicular where it meets the side and will measure 360º/6 in relation to radii from #3.	$\lambda = \frac{\gamma}{2} = 60º$
5) Triangles formed from the center with sides extending to the tips of the polygons are isosceles.	See shaded region.

Determine the analogous properties of a few additional regular polygons in a circle. Refer to diagrams.

Mathematical Properties of a Square in a Circle.	Mathematical Properties of a Pentagon in a circle.	Mathematical Properties of a Hexagon in a circle.
1) $\theta =$	1) $\theta =$	1) $\theta =$
2) $\varphi =$	2) $\varphi =$	2) $\varphi =$
3) $\gamma =$	3) $\gamma =$	3) $\gamma =$
4) $\lambda =$	4) $\lambda =$	4) $\lambda =$
5) Shade in an isosceles triangle.	5) Shade in an isosceles triangle.	5) Shade in an isosceles triangle.
		
6) $\omega =$ [Duh]	6) $\omega =$	6) $\omega =$

Let n be the number of sides of the regular polygon. E.g. For a square, $n = 4$. And let the radius of the circle be one, $r = 1$.

Your homework: Using λ (or ω *or* φ) and *sin* (or *cos*), and x (see diagrams), figure out the general formula for the **perimeter of an <u>inscribed</u> regular polygon**? This will be a formula by which one could choose any number of sides and easily come up with a perimeter. You must use λ (or ω *or* φ) and *sin* (or *cos*), and x as your terms. Write this up so that I can follow your reasoning. This will probably require the drawing of diagrams and some prose.

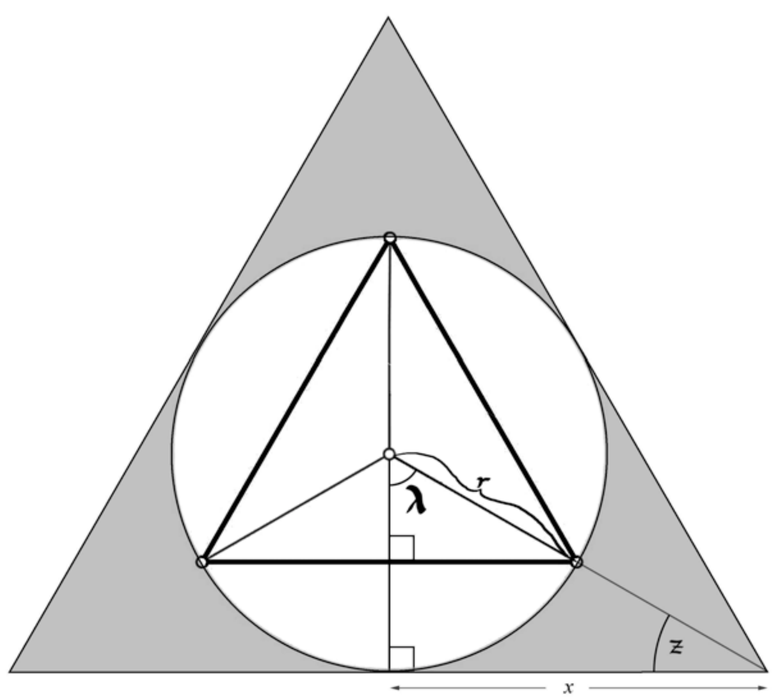
Hint: Look at the patterns in the data from the table on p1. Generalize these patterns into algebraic expressions.

Optional Homework, for those who want a further challenge. (Extra Credit will be awarded.)

1: Come up with a generalized formula for finding the perimeter of any **<u>circumscribed</u> regular polygon**.

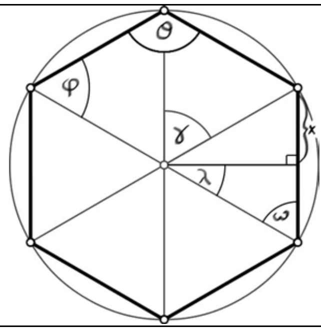
2: The Archimedean π approximation. Once you have a general formula for inscribed regular polygons and one for circumscribed regular polygons you can set about finding a value of π. Choose a polygon to try... say a square. Inscribe a square inside a circle of radius 1 and circumscribe a square around the same circle. Then find their perimeters of each square and π should be in-between. Try to get a value for π that is as close as you can by trying different matching polygons. What is the minimum n necessary to get a value of π accurate to 3 decimal places (i.e. 3.142)?

3: If you really want to be fancy, throw this stuff into Excel or some other computer application and print out some results.



Caveat: There may be a typo or outright mistake on this assignment. I proof-read it many times, but things still slip by. If you find a problem, contact me. Even if there is a mistake, you can hopefully figure out what I meant and continue. If not... contact me. I'm here to help. Feel free to come to office hours or make an appointment.

BLC190-ATF [mifami.org] CLASS 7 SOLUTIONS BASIC GEO-TRIGONO-METRY

	Solution: Using the hexagon as a model on the left and a more generalized approach on the right.
Specifically a Hexagon	Generalization
The perimeter is six times one of the side-lengths. $6 \cdot (sidelength) = perimeter$	Generally the equation will be *n* times the length of a side: $n \cdot (side\ length) = perimeter$, where n is the number of sides of a regular polygon.
The length of a side is twice the length of a half-side (Duh), labeled on the diagram as "x". We can determine *x* using trigonometry. $sin\lambda = \frac{x}{r} = sin30° = \frac{x}{1} =$ $sin30° = x = 0.5$	The length of *x*, a half-side is $sin\lambda = \frac{x}{r}$. And r = 1. So, $sin\lambda = x$. Alright. Fine. But how to write this in generalized terms. I'd like to input the number of sides, *n*, and get a perimeter as output. *P(n)*. So how do I do this?
That means that whole side is *2x = 2(0.5) = 1.*	Well, λ is pretty easy to write in terms of *n*. λ is half of γ and γ is just $\frac{360°}{n}$ and $\lambda = \frac{360°}{2n}$
$6 \cdot (sidelength) = perimeter$ ∴ The perimeter is 6.	So the perimeter of an *n*-sided regular polygon inscribed in a circle of radius 1 is... $Perimeter = n(2x)$ $= 2n(sin\lambda)$ $P_i(n) = 2n\left(sin\frac{360°}{2n}\right)$ $= 2n\left(sin\frac{180°}{n}\right)$ That's your answer.

Test drive it:

$$P(6) = 2(6)\left(sin\frac{180°}{6}\right)$$

$$= 12(sin30°)$$

$$= 12(0.5) = 6$$

Exactly what we hoped for.

So now, let's play with it. What's the ratio of perimeter to diameter? It's $6/2 = 3$. That's hardly a good approximation for π. So let's try a really big value for *n*. How about 100. A 100-sided regular polygon... is almost a circle... sort of... approximately.

$$P(100) = 2(100)\left(sin\frac{180°}{(100)}\right) \cong 6.28215$$

What's the ratio of perimeter to diameter for this 100-sided poly? It's $\frac{6.28215}{2} \cong 3.1412$.

Now that is starting to look like π. It differes from π by 0.000516745776... $\cong$ 5.2×10^{-4}.

$P(1000) \cong 3.141587$ [Differes from π by 0.000005167710... $\cong$ 5.2×10^{-6}]
$P(10000) \cong 3.141593$ [Differes from π by 0.000000051677... $\cong$ 5.2×10^{-8}]

Note: There is an interesting pattern showing up here in the differences between polygon-perimeter/diameter and π. That repeating "5167" is weird. This might be worth another look...or not. But this is a good example of how you follow your nose when exploring math. This type of mathematical curiosity is where Fields Medals come from.

Optional Homework solution: Circumscribed Regular Polygon Formula (for radius 1 circle)

$$P_c(n) = 2n\left(tan\frac{180°}{n}\right)$$

There are other ways to do this using other trigonometric functions, but this seemed the most obvious to me.

If you average circumscribed and inscribed polygons,
you reach an accuracy for π to 3 decimal points (3.142) at about n = 54.

The following table is a section from an Excel spreadsheet.
You can see where Excel punks out with it's approximation of π.
It is only accurate to 15 significant places.
In fact there is a glaring flaw for the very first measurement of perimeter for n = 1. I suspect the reason is that Excel demands angles to be measured in radians, not degrees.
And radians use π, and Excel's π is not very accurate.
Also... What is a polygon with 1 side?... or 2 for that matter?

Perimeter of regular polygon inscribed in a circle of radius 1.

π in Excel = 3.14159265358979000000000000000

General formula: P(n) = 2n(sin(180º/n))
P(n)=2*n*SIN(π/n)

Number of Sides	Perimeter	Perimeter/Diameter	Difference from π
n	**P(n)**	**P(n)/2**	**π – P(n)/2**
1	0.00000000000000024503	0.00000000000000012251	3.14159265358979000000000000000E+00
2	4.00000000000000000000	2.00000000000000000000	1.14159265358979000000000000000E+00
3	5.19615242270663000000	2.59807621135332000000	5.43516442236477000000000000000E-01
4	5.65685424949238000000	2.82842712474619000000	3.13165528843603000000000000000E-01
5	5.87785252292473000000	2.93892626146237000000	2.02666392127427000000000000000E-01
6	6.00000000000000000000	3.00000000000000000000	1.41592653589794000000000000000E-01
7	6.07437234764581000000	3.03718617382291000000	1.04406479766886000000000000000E-01
8	6.12293491784144000000	3.06146745892072000000	8.01251946690749000000000000000E-02
9	6.15636257986204000000	3.07818128993102000000	6.34113636587745000000000000000E-02
10	6.18033988749895000000	3.09016994374947000000	5.14227098403190000000000000000E-02
20	6.25737860160923000000	3.12868930080462000000	1.29033527851758000000000000000E-02
30	6.27170779605921000000	3.13585389802960000000	5.73875556018955000000000000000E-03
40	6.27672765822760000000	3.13836382911380000000	3.22882447599548000000000000000E-03
50	6.27905195293134000000	3.13952597646567000000	2.06667712412445000000000000000E-03
60	6.28031474915326000000	3.14015737457663000000	1.43527901316354000000000000000E-03
70	6.28107624907209000000	**3.14053812453604000000**	1.05452905374825000000000000000E-03
80	6.28157052145098000000	3.14078526072549000000	8.07392864304202000000000000000E-04
90	6.28190940645017000000	3.14095470322509000000	6.37950364705819000000000000000E-04
100	6.28215181562566000000	3.14107590781283000000	5.16745776963923000000000000000E-04
1,000	6.28317497175913000000	**3.14158748587956000000**	5.16771022951446000000000000000E-06
10,000	6.28318520382533000000	3.14159260191267000000	5.16771279102102000000000000000E-08
100,000	6.28318530614604000000	3.14159265307302000000	5.16771514469383000000000000000E-10
1,000,000	6.28318530716925000000	3.14159265358463000000	5.16742204581533000000000000000E-12
10,000,000	6.28318530717948000000	3.14159265358974000000	5.19584375524573000000000000000E-14
100,000,000	6.28318530717958000000	3.14159265358979000000	0.00000000000000000000000000000E+00
1,000,000,000	6.28318530717959000000	3.14159265358979000000	0.00000000000000000000000000000E+00
10,000,000,000	6.28318530717959000000	3.14159265358979000000	0.00000000000000000000000000000E+00