Homework 5
Solutions

Exercise 1: USING EQ. 1. $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$

1. Find the total amount you will have if you invest \$100, at 8% (annual rate), compounded **annually** for …

a) 1 year	$A(t) = \$100\left(1+\frac{0.08}{1}\right)^1 = 100(1.08) = \$108.00$
b) 10 years	$A(10) = \$100\left(1+\frac{0.08}{1}\right)^{10} \cong \$215.892$
c) 30 years	$A(30) = \$100(1.08)^{30} \cong \$1,006.266$

2. Find the total amount you will have if you invest \$100, at 8% (annual rate), compounded **seasonally** for …
I did part b to show you how.

a) 1 year	$A(1) = \$100\left(1+\frac{0.08}{4}\right)^{4\cdot 1} \cong \$108.243$
b) 10 years	$A(10) = \$100\left(1+\frac{0.08}{4}\right)^{4\cdot 10} = \$100(1.02)^{40} \cong \$220.80.$ You more than doubled your money in 10 years.
c) 30 years	$A(30) \Rightarrow \cong \$1076.52$

3. Find the total amount you will have if you invest \$100, at 8% (annual rate), compounded **daily** for …

I chose 365 but 365.25 can work too.

a) 1 year	$A(1) = \$100\left(1+\frac{0.08}{365}\right)^{365\cdot 1} = \$100\left(1+\frac{0.08}{365}\right)^{365} \cong \$108.33$
b) 10 years	$A(10) = \$100\left(1+\frac{0.08}{365}\right)^{3650} \cong \$222.54$
c) 30 years	$A(30) \cong \$1102.03$

4. Your daughter is 7-years old and you really want her to have some money for college when she is 17. So you take $10,000 and give it to your investment manager, Lefty Luciano, and ask him what he can do. He gives you a choice. He'll invest it at 7.2% compounded annually or he'll invest it at 7% compounded daily. What should you do? [Figure out how much you will have after 10 years.]

$t = 10$

$P = 10{,}000$

Annually

$r = 7.2\% = 0.072$

$n = 1$

$A(10) = 10{,}000(1 + 0.072)^{10}$

$= \$20{,}042.31$

Daily

$r = 7.0\% = 0.07$

$n = 365$

$A(10) = 10{,}000\left(1 + \frac{0.07}{365}\right)^{365 \cdot 10}$

$= \$20{,}136.18$

The winner!

---

5. It's the year 2007. You have $80,000 in credit card debt, at 15% annual interest, compounded monthly. The banks have shut you down at $80,000. You can't borrow any more. But you could care less. Spending that $80k was a lot of fun. So you move to Venezuela to escape your debts. *Hasta la vista los bancos de Los Estatos Unitos. Ojala que te den por ...etc. Joder! Viva Hugo!* Venezuela is one great big super-party and Hugo Chavez is the DJ. But then two things happen. Oil prices tank and Hugo "DJ" Chavez dies. The party is over and the subsequent depression makes North Korea's economy look appealing.

After 10 years of living in turbulent Venezuela, you can't take it any more. So you start thinking about returning to the U.S. But you owe all that money to the bank. It's been 10 years. How much could it be?

Write this up with the appropriate formula. Show what all the parts of the formula mean and crank out a final dollar amount. How much will it ultimately cost you to return to America, assuming you want to live a normal life with a passport and a bank account?

$P = \$80{,}000$

$r = 15\% = 0.15$

$n = 12$ (monthly)

$t = 10$ years

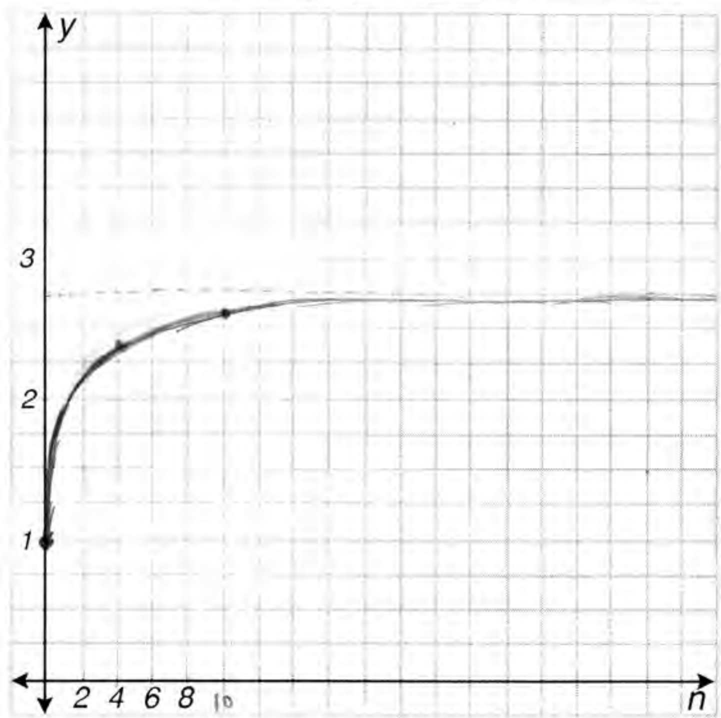
$A(10) = 80{,}000\left(1 + \frac{0.15}{12}\right)^{12(10)} = 80{,}000(1.0125)^{120} =$

$= \$355{,}217.06$ OUCH!

You notice that part of Eq. 1 (above) involves $\left(1+\frac{1}{n}\right)^n$. Let's play with this part.

Exercise 2: Solve for y in the following equation by inputting n: $y=\left(1+\frac{1}{n}\right)^n$. Then graph it.

n =	**y =**
e.g. 1	2
2	2.25
4	~2.44
10	~2.59
100,000	~2.72
lim $n \to \infty$	e
0.5	~1.73
0.1	~1.27
0.001	~1.01



$\boldsymbol{e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=2.718...}$ This is sometimes referred to as Euler's Number. [Pronounced *oiler*.]
This is another π, another φ, another one of those weird, irrational places in the world of numbers.
This is e.

Now recall Eq. 1.

Equation 1 deals with compound interest. **P**rincipal, interest **r**ate, **n**umber of compoundings, and **t**ime/duration.	Eq.1 $A(t)=P\left(1+\frac{r}{n}\right)^{nt}$
Just playing with parentheses here. Nothing substantially changed.	$A(t)=P\left[\left(1+\frac{r}{n}\right)^n\right]^t$
Now being fancy with the exponents. $nt=\frac{n}{r}\cdot rt=nt.$	$A(t)=P\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$
Make up a new variable, call it, $m=\frac{n}{r}$. And so $\frac{1}{m}=\frac{r}{n}$. Sub these in... OMG! Look! $\left(1+\frac{1}{m}\right)^m$ That's related to e.	$A(t)=P\left[\left(\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{m}}\right)^{\boldsymbol{m}}\right]^{rt}$
If we allow... $e=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m=2.718...$ ...then...	$A(t)=Pe^{rt}$ **Eq. 2**
That's like saying that we're going to compound the interest in Eq. 1 an infinite number of times... in other words, *continuously*.	$\lim_{n\to\infty}(P)\left(1+\frac{r}{n}\right)^{nt}$

$$\text{Eq. 2} \qquad A(t) = Pe^{rt}$$

Just remember, the rate and the time need to match in terms of their time periods.
E.g. $r = 12\%$ per month. And $t = 12$ months, NOT $t = 1$ year.

This equation is the basis for all sorts of growth models including disease modeling, bacterial reproduction, radioactive decay, and of course compound interest. Let's get to work!

Exercise 3. Find the total amount you will have if you invest $100, at 8% (annual rate), compounded **continuously** for the following times. [I'll do part b to show you what's what.]

a) 1 year $A(1) = \$100e^{0.08(1)} \cong \$108.33$
b) 10 years $A(t) = Pe^{rt}$ $A(10) = \$100e^{0.08 \cdot 10} \cong \$222.55$ [Your calculator probably has an "e" button.]
c) 30 years $A(30) = \$100\,e^{0.08(30)} \cong \$1102.32$

Exercise 4. At 1:00 pm a bacterial culture taken from a patient's throat [*Streptococcus pyogenes*] contains 20 bacteria. At 4:00 pm the population has risen to 78.

a) Assuming exponential growth (use Eq. 2), find the growth rate. [Population increase per hour.] Hint: A=78, P=20, t=3. Solve for r. $78 = 20e^{r(3)} \Rightarrow e^{3r} = \frac{78}{20}$ $\quad \ln e^{3r} = \ln \frac{78}{20}$ $3r \cong 1.36 \rightarrow r \cong 0.45/\text{hr}$
b) Find a general formula for the population after t hours. $A(t) = 20e^{0.45t}$
c) How many bacteria are there at 9:00 pm? $A(8) = 20e^{0.45(8)} \cong 731.96$
d) How many in 1 full day? $A(24) \cong 980{,}416 \cong 9.8 \times 10^5$
e) How long (in hours:minutes:seconds) will it take for the population to hit 10,000,000? " 10^7 $10^7 = 20\,e^{0.45(t)} \Rightarrow \ln \frac{10^7}{20} = \ln e^{0.45t}$

$\Downarrow$

$0.45t \cong 13.12$

$t \cong 29.16 \text{ hrs} \cong 29^{\text{hrs}}\, 9'30''$

How to use you a rock as a clock. Put a large rock in your oven at 500°. You could define a period of time as *how long it takes that rock to cool off to 250°*. There's your clock. A hot rock. Because of fluctuations in room temperature and other uncontrollable factors, the amount of time it takes your rock to cool off to 0° varies significantly. For example, if the ambient temperature is 15° your rock will never cool off to 0°. But the time for your rock to cool to 250° will not vary all that much.

Radio Carbon Dating: The half-life of ^{14}C is about 5730 years. The ratio of $^{14}C:^{12}C$ in a fresh plant is approximately 1:1,000,000. [One atom of ^{14}C per 1 million atoms of ^{12}C.] You are given a sample of wood from a beam from an archeological site. The measurement of its carbon isotopes yields a ratio of $^{14}C:^{12}C$ is 0.5:1,000,000. [One half parts per million.] In other words, its ^{14}C content is one half of what a freshly cut piece of wood would have.

So let's put all this data into our exponential equation and solve for the decay rate. The resultant amount is 1/2. The initial amount is 1. And the time is 5730 years. [Because that's the definition of half-life. Half of the ^{14}C remains.]	$A(t) = Pe^{rt}$ $1/2 = 1e^{5730r}$
Solve for the rate of decay by rewriting into log form.... $x = b^y$ is $y = \log_b x$. Then just solve for r.	$1/2 = e^{5730r}$ $\ln 1/2 = 5730r$
That's the rate of decay. It's in the exponent of the compound interest formula (Eq. 2). It's negative because it diminishes quantities instead of growing them. That negative sign puts e^{rt} in the denominator. It's now a fraction.	$r \cong -0.000121$ $\cong -1.21 \times 10^{-4}$
Here is the radioactive decay formula for ^{14}C:	**Eq.3** $A(t) = Pe^{-0.000121t}$

Exercise 5: Given the ratios of $^{14}C:^{12}C$, figure out how old the samples are.

a) You are given a desiccated tissue sample from Siberia which yields a ratio of ($0.12\,^{14}C$):(1million ^{12}C).

$A(t) = Pe^{-0.000121(t)}$

$0.12 = (1)e^{-0.000121t}$

$\ln 0.12 = \ln e^{-0.000121t}$

$-2.12 \cong -0.000121t \Rightarrow t \cong 17,522.8$ years

Hint: A is 0.12, P is 1 and the answer's first two digits are 1 and 7.

b) You are given fragments of a papyrus scroll found in Egypt. The measured ratio is 0.7:1million.

$0.7 = e^{-.000121t}$

$\ln 0.7 = -0.000121t \rightarrow t = 2947.7$ years

Hint: third digit is 4.

c) You are given some bone fragments from an unknown animal. The measured ratio is 0.002:1million. [This ratio is about as small as can be accurately measured. Carbon 14 dating is unreliable much beyond this.]

$0.002 = e^{-0.000121t} \longrightarrow t \cong 51,360.4$ years

Hint: first digit is 5.

d) Write up an equation, based on Eq. 3, which solves for t. In other words, solve for t in Eq. 3 so that you can easily input ^{14}C amounts and get back years old.

$t = \dfrac{\ln(^{14}C)}{-0.000121}$ where ^{14}C = amount of ^{14}C from baseline of 1.

Here is a website you can use to check your answers: https://www.math.upenn.edu/~deturck/m170/c14/carbdate.html

After about 55,000 years, Radio Carbon Dating is not useful. Then you move on to Uranium–thorium dating which can go back about 500,000 years. Then to Uranium–lead dating which has a range of about 1 million to 4.5 billion years. Or, if you really want to go retro, use Rubidium–strontium dating which can take you back more than 50 billion years.... well beyond the Big Bang. The general concepts are all the same– radioactive decay. Stuff decays off at a constant rate (it is assumed). Much like how you could use a rock as a clock. You are just measuring how long it takes to metaphorically cool off.

Exercise 6: Just do some math. Solve for the variable.

E.g. $\ln(3x) = 10$ $\ln(3) + \ln(x) = 10$ $\ln(x) = 10 - \ln(3) \cong 8.901$ $e^{8.901} = x \cong 7342.16$	a) $\ln(16x) = 3$ $\ln 16 + \ln x = 3$ $\ln x \cong 0.227$ $e^{0.227} \cong x \cong 1.25$
b) $\log_4 x + \log_4(x-6) = 2$ $\log_4 x(x-6) = 2$ $\log_4(x^2-6x) = 2$ rewrite as exp $4^2 = x^2 - 6x$ $x^2 - 6x - 16 = 0$ $(x-8)(x+2) \rightarrow (8, -2)$ Hint: Use log product property (Eq. 2a from HW-4) and then some basic factoring.	c) $e^{2x} - 2e^x - 8 = 0$ $(e^x - 4)(e^x + 2) = 0$ $e^x = 4 \Rightarrow \ln 4 = x \cong 1.39$ $e^x = -2 \Rightarrow \ln(-2) = x \cong$ Not in domain
d) $5^x = 12.328$ $\log_5 12.328 = x = \frac{\ln 12.328}{\ln 5} \cong 1.56$	e) $4e^{2x} = 20e$ $\frac{e^{2x}}{e} = 5$ $e^{2x-1} = 5$ $\ln e^{2x-1} = \ln 5$ $2x - 1 \cong 1.61$ $x \cong 1.30$
f) $e^{4x} - 7e^{2x} + 12 = 0$ $(e^{2x} - 4)(e^{2x} - 3) = 0$ $e^{2x} = 3$ $x \cong 0.55$ $e^{2x} = 4$ $2x = \ln 4$ $x \cong 0.69$	Take a break. Thank you.

Extra Credit: Wind Chill: Maurice Bluestein (a mechanical engineer) was shoveling snow one evening in 1994. It was -25°F. The Weather Service, using an old formula, declared that it felt like it was -65°F. This effective temperature would supposedly cause frostbite in 15 seconds. Dr. Bluestein was dubious. He didn't feel very cold. He had been outside for much longer than 15 seconds and had not frozen in place. In fact, with all the shoveling, he was hot. Several years later he came up with what we now use to determine the **W**ind **C**hill **T**emperatures.

$$\boldsymbol{WTC = 35.74 + 0.6215T - 35.75V^{0.16} + 0.4275TV^{0.16}}$$ [T = °F, and V = wind speed (mph)]

Given T = -25°F, and a modern WTC = -65°F, what would be the wind speed? [Solve for V.] Show work.

$-65 = 35.74 + (0.6215)(-25) - 35.75V^{0.16} + 0.4275(-25)V^{0.16}$

$-65 = 35.74 - 15.5375 - 35.75V^{0.16} - 10.6875V^{0.16}$

$-85.2025 = V^{0.16}(-46.4375)$

$V^{0.16} \cong 1.835$

rewrite as exponential

$\log_V 1.835 = 0.16$

$\frac{\ln 1.835}{\ln V} = 0.16$

$\ln V = \frac{\ln 1.835}{0.16} \cong 3.79$

$\ln V = 3.79$

$e^{3.79} = V \cong 44\text{ mph}$