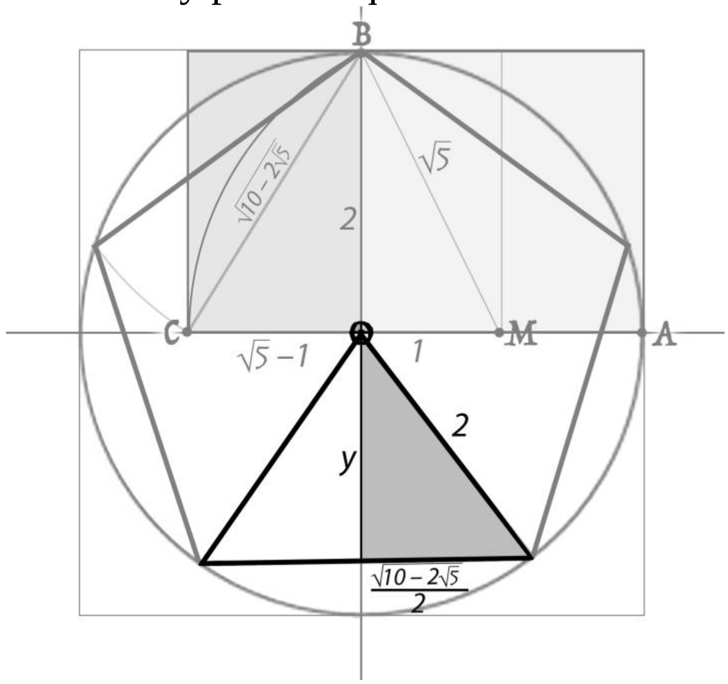
BLC190- ATF-Spring 2020

Homework 10: Tricky perfect squares and more Geometric Trig.



When making the regular pentagon (see videos) we ended up with the following for the length of the side of the regular pentagon in a circle of radius 2.

$$\sqrt{10-2\sqrt{5}}$$

Focus on the darkest shaded right triangle above.

The short leg is $\frac{\sqrt{10-2\sqrt{5}}}{2}$,

half of the length of a side of the pentagon.

The hypotenuse is 2, ...the radius of the circle.

What is the long leg?

Just employ the Pythagorean Theorem.

$$\left(\frac{\sqrt{10-2\sqrt{5}}}{2}\right)^2 + y^2 = 2^2$$

$$= \frac{10-2\sqrt{5}}{4} + y^2 = 4$$

$$y^2 = \quad 4 - \frac{10-2\sqrt{5}}{4} \quad = \quad \frac{16}{4} - \frac{10-2\sqrt{5}}{4}$$

$$y^2 = \frac{16-10+2\sqrt{5}}{4}$$

$$y^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} \quad \rightarrow$$

$$\boldsymbol{y = \sqrt{\frac{3+\sqrt{5}}{2}} \cong 1.618} \ (\textit{\textbf{approx. from calculator}})$$

I recognize this number! 1.618. *It is the golden ratio*, φ!

But I know from my studies of the golden ratio that

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

How do I turn $\sqrt{\frac{3+\sqrt{5}}{2}}$ *into* $\frac{1+\sqrt{5}}{2}$.

It's relatively easily to prove that $\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$.

$$\textit{Square both sides ...} \quad \frac{3+\sqrt{5}}{2} = \frac{1+2\sqrt{5}+5}{4}$$

$$\rightarrow \quad \frac{6+2\sqrt{5}}{4} = \frac{6+2\sqrt{5}}{4} \quad \textit{Bingo}!$$

But I would never have known that

$$\sqrt{\frac{3+\sqrt{5}}{2}}$$

was the golden ratio if I had not recognized the number 1.618... on my calculator screen.

Question: How do I simplify $\sqrt{\frac{3+\sqrt{5}}{2}}$?

How do I get that square root
out from under that square root?

You somehow need to turn $\frac{3+\sqrt{5}}{2}$ into a squared thing,

so that when you "unsquare" it it comes out clean...so

that $\sqrt{x^2} = x$.

A squared polynomial thing should look like this:

$a^2 + 2ab + b^2 = (a+b)^2$.

A completed square quadratic.

But we have this: $\frac{3+\sqrt{5}}{2}$.

How do we make $\frac{3+\sqrt{5}}{2} = (a+b)^2$

Here's the trick that will help you see how this can be done. Multiply top and bottom by 2.

$$\frac{6+2\sqrt{5}}{4}$$

Now it is starting to look like $a^2 + 2ab + b^2$. Sort of.

We have something like a *2ab*.... $2\sqrt{5}$.

Now split that 6 into 5 + 1 and rearrange.

$$\frac{5+2\sqrt{5}+1}{4}.$$

Now it looks a lot more like $a^2 + 2ab + b^2$.

Now rewrite it as a perfect square...

$$\frac{\left(\sqrt{5}+1\right)^2}{2^2}$$

We are basically just completing the square.

The question was... How do I simplify $\sqrt{\frac{3+\sqrt{5}}{2}}$?

$$\sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\frac{\left(\sqrt{5}+1\right)^2}{2^2}} = \frac{\mathbf{\sqrt{5}+1}}{\mathbf{2}}.$$

The square root of a square.

That's how I want to see the Golden Ratio.

$$\boldsymbol{\varphi} = \frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}} \cong \mathbf{1.618}$$

That's a long mathematical trip just to get a familiar notation.

The larger point was to demonstrate this way of completing a square.

Exercise

The point is to get the root sign out from under the root sign. That's just ugly math.
I don't want decimal approximations. I want pure simplifications like the one shown above.
Once you notice the pattern after doing 2 or 3 of them the rest will be super easy. Just look at the example and figure out the method. Keep in mind, this trick only works for a very specific type of expression.

Simplify $\sqrt{3-2\sqrt{2}}$. $$\sqrt{2-2\sqrt{2}+1}=\sqrt{\left(\sqrt{2}-1\right)^2}=\sqrt{2}-1$$
Simplify $\sqrt{3+2\sqrt{2}}$.
Simplify $\sqrt{8+2\sqrt{7}}$
Simplify $\sqrt{8-2\sqrt{7}}$
Simplify $\sqrt{12+2\sqrt{11}}$
Simplify $\sqrt{12-2\sqrt{11}}$
Simplify $\sqrt{34+2\sqrt{33}}$
Simplify $\sqrt{34-2\sqrt{33}}$
Simplify $\sqrt{6+2\sqrt{5}}$
Simplify $\sqrt{6-2\sqrt{5}}$
Try to Simplify $\sqrt{10+2\sqrt{5}}$. [You probably won't be able to do this one. If you can, let me know.]

Sine, Cosine, and Tangent

(See Precalculus Review Notes on Website for additional details)

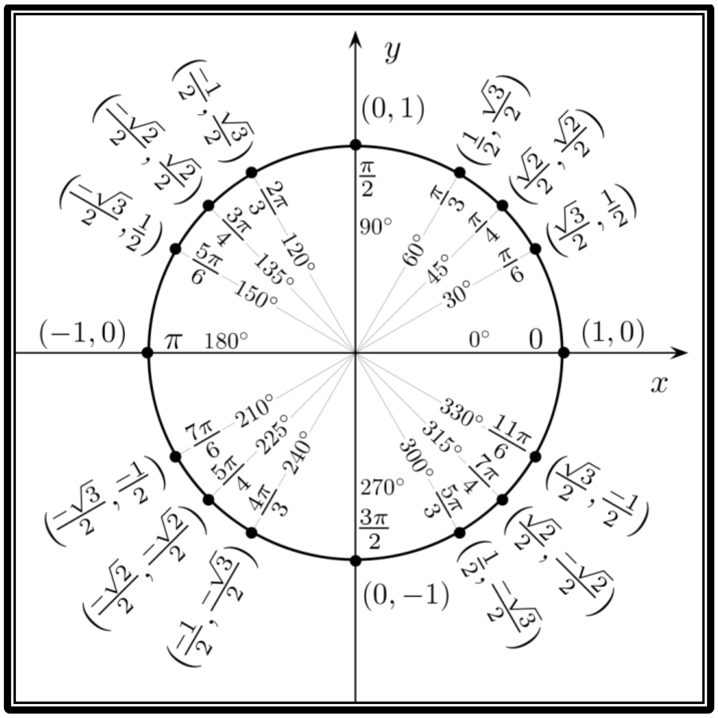
$$sin = \frac{opposite}{hypotenuse} \qquad cos = \frac{adjacent}{hypotenuse} \qquad tan = \frac{sin}{cos} = \frac{\frac{opposite}{hypotenuse}}{\frac{adjacent}{hypotenuse}} = \frac{opposite}{adjacent}$$

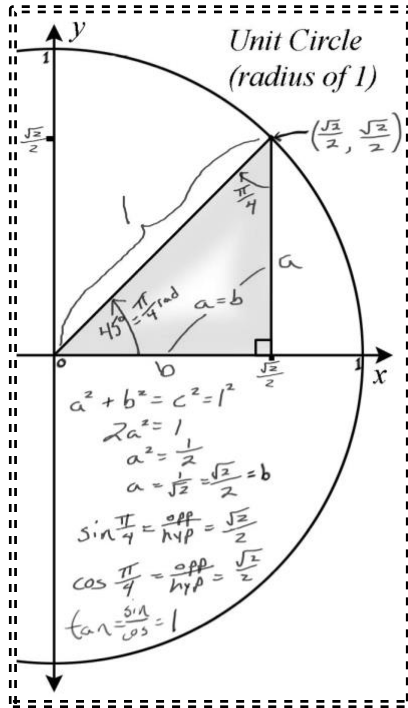
csc, *sec*, and *cot* are just the reciprocal functions of *sin*, *cos*, and *tan* respectively: *1/sin*, *1/cos*, *1/tan*.

A way to remember the sine of commonly encountered angles:

$\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$ are the sine of 0º, 30º, 45º, 60º, 90º *or* $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$

The cosine is just reversed.





The chart on the left shows the *cosy* and *siny*, i.e. (*x*, *y*) = (*cosy*, *siny*), in terms of degrees and radians on a unit circle (*r* = 1). The diagram on the right is a detail of the right half of a **unit circle** and it focuses on π/4 or 45º and shows how the Pythagorean Theorem is involved in the determination of sines and cosines.

Trigonometry (triangle measurement) is the Pythagorean Theorem turned into algebra.

The x-axis is cosine because it is measuring the adjacent leg of the right triangle.

Trigonometric Angles and the basic operations: Addition, Subtraction, Double, and Half

Angle Addition formulas [We'll derive these in class] $\mathbf{sin}(\boldsymbol{\theta}+\boldsymbol{\gamma}) = \boldsymbol{sin\theta cos\gamma} + \boldsymbol{sin\gamma cos\theta}$ $\mathbf{cos}(\boldsymbol{\theta}+\boldsymbol{\gamma}) = \boldsymbol{cos\theta cos\gamma} - \boldsymbol{sin\theta sin\gamma}$	Angle Subtraction formulas [We didn't derive these, but the derivation is similar.] $\mathbf{sin}(\boldsymbol{\theta}-\boldsymbol{\gamma}) = \boldsymbol{sin\theta cos\gamma} - \boldsymbol{sin\gamma cos\theta}$ $\mathbf{cos}(\boldsymbol{\theta}-\boldsymbol{\gamma}) = \boldsymbol{cos\theta cos\gamma} + \boldsymbol{sin\theta sin\gamma}$

Double Angle Formulas $[\theta = \gamma]$

$$\mathbf{sin}(\mathbf{2}\boldsymbol{\theta}) = \mathbf{2}\boldsymbol{sin\theta cos\theta}$$

$$\mathbf{cos}(\mathbf{2}\boldsymbol{\theta}) = \mathbf{2}\boldsymbol{cos^2\theta} - \mathbf{1}$$

$$\mathbf{cos}(\mathbf{2}\boldsymbol{\theta}) = \mathbf{1} - \mathbf{2}\boldsymbol{sin^2\theta}$$

Half Angle Formulas

[derived from Double Angle Formulas]

$$cos\gamma = cos\left[2\left(\frac{\gamma}{2}\right)\right] = 2cos^2\left(\frac{\gamma}{2}\right) - 1$$

$$\frac{1+cos\gamma}{2} = cos^2\left(\frac{\gamma}{2}\right) \quad \rightarrow \quad \boldsymbol{cos}\left(\frac{\boldsymbol{\gamma}}{\mathbf{2}}\right) = \sqrt{\frac{\mathbf{1}+\boldsymbol{cos\gamma}}{\mathbf{2}}}$$

Similarly....

$$cos\gamma = cos\left[2\left(\frac{\gamma}{2}\right)\right] = 1 - 2sin^2\left(\frac{\gamma}{2}\right)$$

$$\frac{cos\gamma - 1}{2} = -sin^2\left(\frac{\gamma}{2}\right) \quad \rightarrow \quad \boldsymbol{sin}\left(\frac{\boldsymbol{\gamma}}{\mathbf{2}}\right) = \sqrt{\frac{\mathbf{1}-\boldsymbol{cos\gamma}}{\mathbf{2}}}$$

Table of Known Values

All hypotenuses are equal to 1, which is the same as saying, "on the unit circle."

Angle in π Radians	Angle in Radians	Angle in Degrees	Derived from...	Sine	Cosine	Tangent
0π	0	0º	n.a.	0	1	
0.16667π	0.52359...	30º	Equilateral	0.5	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
0.2π	0.62831...	36º	Pentagonal section	0.587785252 $= \frac{\sqrt{10-2\sqrt{5}}}{4}$	0.809016994 $= \frac{\varphi}{2}$	0.72654...
0.25π	0.78539...	45º	Right Isosceles	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
0.3π	0.94247...	54º	Pentagonal section	0.809016994 $= \frac{\varphi}{2}$	0.587785252 $= \frac{\sqrt{10-2\sqrt{5}}}{4}$	1.37638...
0.33333π	1.04719...	60º	Equilateral	$\frac{\sqrt{3}}{2}$	0.5	1.73205...
0.5π	1.57079...	90º	n.a.	1	0	und.
π	3.1415...	180º	n.a.	0	-1	0

Sines, Cosines, and Tangents derived from the geometry of known triangles.

Exercise

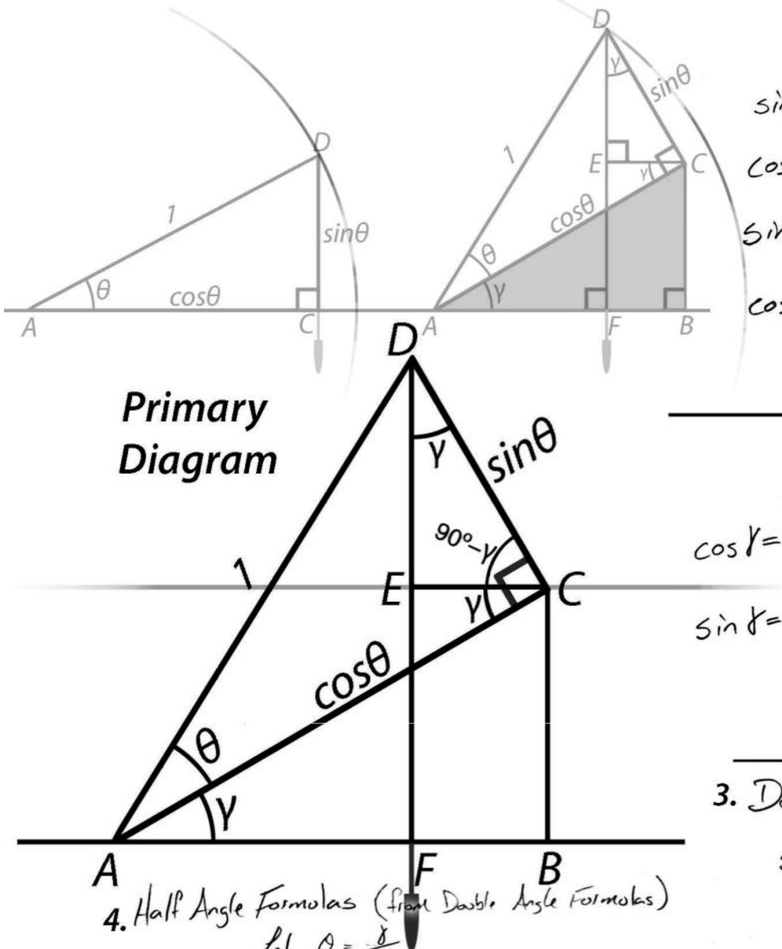
Using the Addition, Subtraction, Double, and Half Angle Formulas and the table of known values above, construct a table of sines, cosines, and tangents. Fill up the rest of the table with angles of your choosing. Feel free to use symmetry across the 45º angle for justifications. There are multiple ways to arrive at most of these. Use whatever works. I suggest an Excel spreadsheet.

Just don't just type the angle into a calculator and hit "sin". Use the +, –, double, and half angle formulas. These can be time consuming.

Everybody should do the shaded rows. Then do 2 more rows of your choice.

	Sine	Cosine	Tangent	Combination used to get result and/or justification.
0º				We know this already. Just copy it in.
3º				
6º				Double Angle on 3º
9º				Try 45º – 36º
12º				Try 30º – 18º
15º				Half angle applied to 30º.
18º				Try Half-Angle on 36º.
21º				
24º				
27º				
30º				We know this already. Just copy it in.
33º				Try 15º + 18º
36º				We know this already. Just copy it in.
39º				
42º				
45º				We know this already. Just copy it in.
48º				Sine and Cosine switched from 42º due to symmetry across 45º
54º	$= \frac{\varphi}{2}$	$= \frac{\sqrt{10 - 2\sqrt{5}}}{4}$		We know this already. Just copy it in.

We will discuss this in class. Feel free to look it over.



$\sin\theta = \frac{\overline{DC}}{1} = \overline{DC}$

$\cos\theta = \frac{\overline{AC}}{1} = \overline{AC}$

$\sin\gamma = \frac{\overline{CB}}{\cos\theta} \rightarrow \overline{CB} = \sin\gamma\cos\theta$

$\cos\gamma = \frac{\overline{DE}}{\sin\theta} \rightarrow \overline{DE} = \sin\theta\cos\gamma$

1. Sine Angle Addition

$\sin(\theta+\gamma) = \frac{\overline{DF}}{1} = \overline{DF}$

$\overline{DF} = \overline{DE} + \overline{EF}$

$\overline{DF} = \overline{DE} + \overline{CB}$

$$\boxed{\sin(\theta+\gamma) = \sin\theta\cos\gamma + \sin\gamma\cos\theta}$$

$\cos\gamma = \frac{\overline{AB}}{\cos\theta} \rightarrow \overline{AB} = \cos\theta\cos\gamma$

$\sin\gamma = \frac{\overline{EC}}{\sin\theta} \rightarrow \overline{EC} = \sin\theta\cos\theta$

2. Cosine Angle Addition

$\cos(\theta+\gamma) = \frac{\overline{AF}}{1} = \overline{AF}$

$\overline{AF} = \overline{AB} - \overline{FB}$ $\quad [\overline{FB} = \overline{EC}]$

$$\boxed{\cos(\theta+\gamma) = \cos\theta\cos\gamma - \sin\theta\sin\gamma}$$

3. Double Angle Formulas → set $\theta = \gamma$

$\sin(\theta+\theta) = \sin(2\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$$

$\cos(\theta+\theta) = \cos(2\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

[Recall that $\sin^2\theta + \cos^2\theta = 1$ … so either $\cos^2\theta = 1 - \sin^2\theta$ or $\sin^2\theta = 1 - \cos^2\theta$]

$\cos(2\theta) = \cos^2\theta - 1 + \cos^2\theta$

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1}$$

…or…

$\cos(2\theta) = 1 - \sin^2\theta - \sin^2\theta$

$$\boxed{\cos(2\theta) = 1 - 2\sin^2\theta}$$

4. Half Angle Formulas (from Double Angle Formulas)

Let $\theta = \frac{\gamma}{2}$

$\cos\left(2\frac{\gamma}{2}\right) = \cos\gamma = 2\cos^2\left(\frac{\gamma}{2}\right) - 1$

$\cos^2\left(\frac{\gamma}{2}\right) = \frac{1+\cos\gamma}{2}$

$$\boxed{\cos\frac{\gamma}{2} = \sqrt{\frac{1+\cos\gamma}{2}}}$$

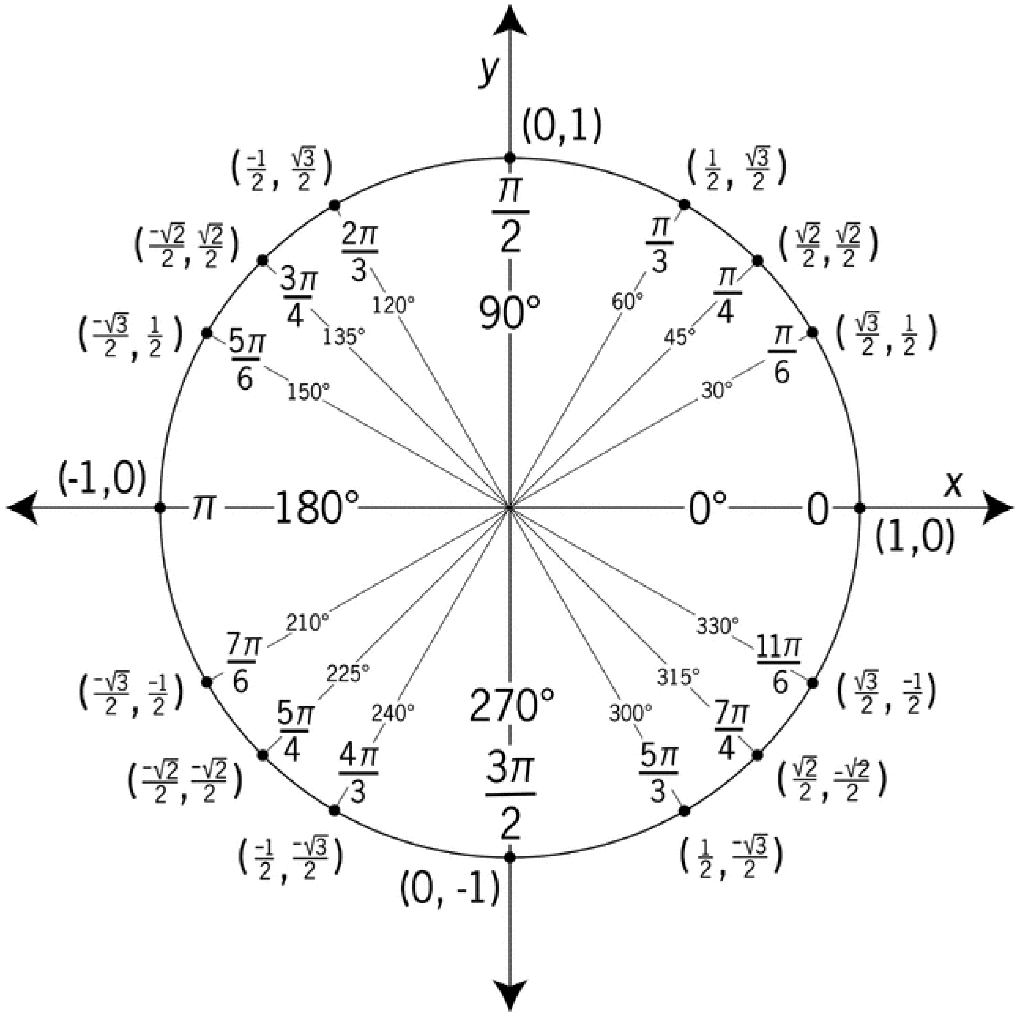
$\cos\left(2\frac{\gamma}{2}\right) = \cos\gamma = 1 - \sin^2\left(\frac{\gamma}{2}\right)$

$\sin^2\left(\frac{\gamma}{2}\right) = \frac{1-\cos\gamma}{2}$

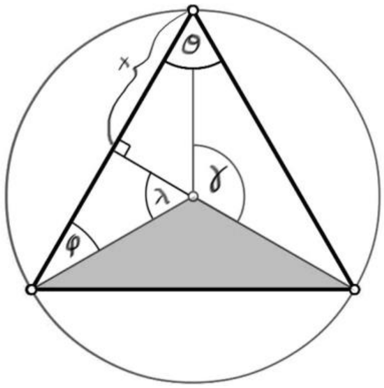
$$\boxed{\sin\left(\frac{\gamma}{2}\right) = \sqrt{\frac{1-\cos\gamma}{2}}}$$

The Unit Circle. $(cos\theta, sin\theta)$

(The "Unit Circle" has a radius of 1... unit... as in unicycle, unison, universe, etc.)



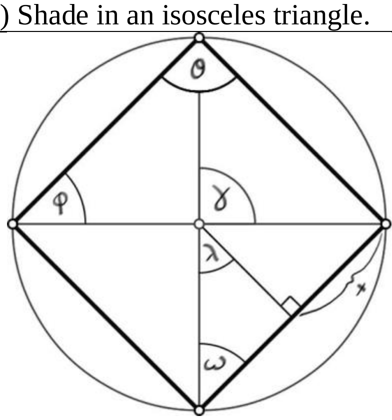
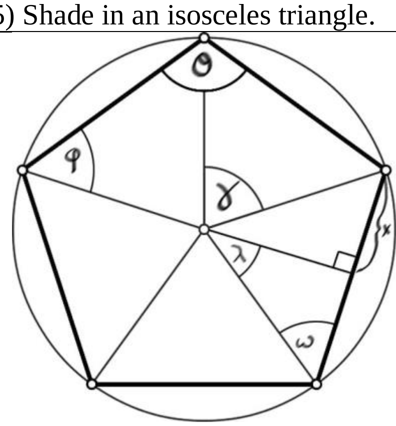
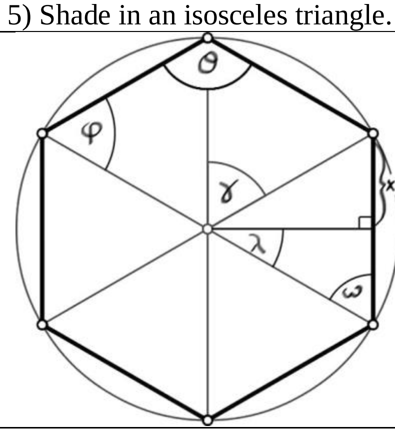
Everything below is Optional:



Some Mathematics Pertaining to Regular Polygons Inscribed in a Circle

A Few Mathematical Properties of an Equilateral Triangle Inscribed in a Circle	Example: equilateral triangle (see above)
1) All interior angles of a **triangle** add up to 180º.	$\theta = 60º$
2) Interior angles (θ) are bisected by radii drawn from the center of the circle to a vertex.	$\varphi = \frac{\theta}{2} = \frac{180º - \gamma}{2} = 30º$
3) Angles between rays γ which extend to vertices measure 360º/3. [Note: γ = γ]	γ = 120º
4) Angles formed by drawing a radius to the middle point of any side of the triangle will be perpendicular where it meets the side and will measure 360º/6 in relation to radii from #3.	$\lambda = \frac{\gamma}{2} = 60º$
5) Triangles formed from the center with sides extending to the tips of the polygons are isosceles.	See shaded region.

Determine the analogous properties of a few additional regular polygons in a circle. Refer to diagrams.

Mathematical Properties of a Square in a Circle.	Mathematical Properties of a Pentagon in a circle.	Mathematical Properties of a Hexagon in a circle.
1) $\theta =$	1) $\theta =$	1) $\theta =$
2) $\varphi =$	2) $\varphi =$	2) $\varphi =$
3) $\gamma =$	3) $\gamma =$	3) $\gamma =$
4) $\lambda =$	4) $\lambda =$	4) $\lambda =$
5) Shade in an isosceles triangle.	5) Shade in an isosceles triangle.	5) Shade in an isosceles triangle.
		
6) $\omega =$ [Duh]	6) $\omega =$	6) $\omega =$

Let n be the number of sides of the regular polygon. E.g. For a square, $n = 4$. And let the radius of the circle be one, $r = 1$.

Your homework: Using λ (or ω *or* φ) and *sin* (or *cos or tan*), and x (see diagrams from table above), figure out the **general formula** for the **perimeter of an <u>inscribed</u> regular polygon**? This will be a formula by which one could choose any number of sides and easily come up with a perimeter. You must use λ (or ω *or* φ) and *sin* (or *cos or tan*), and x as your terms. Write this up so that I can follow your reasoning. This will probably require the drawing of diagrams and some prose.

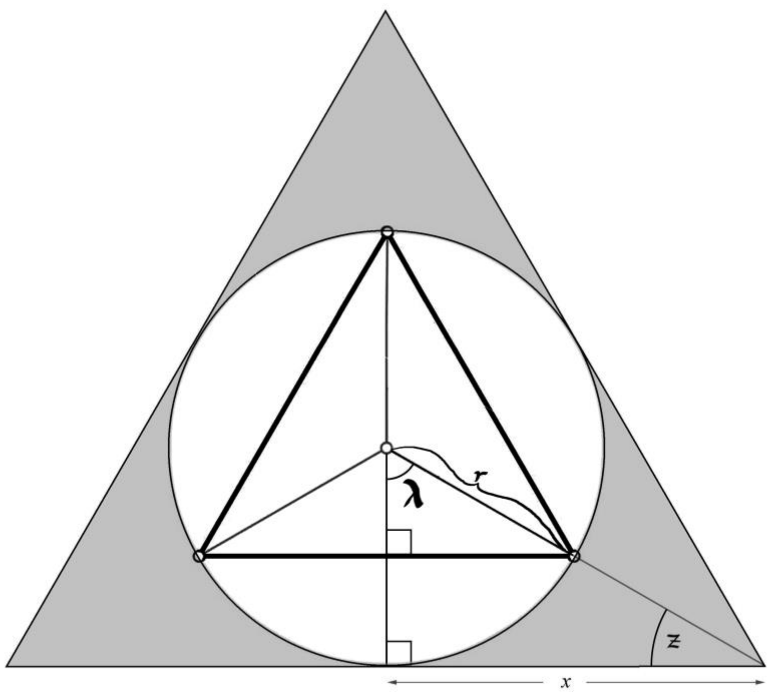
Hint: Look at the patterns in the data from the table on p1. Generalize these patterns into algebraic expressions.

Optional Homework, for those who want a further challenge. (Extra Credit will be awarded.)

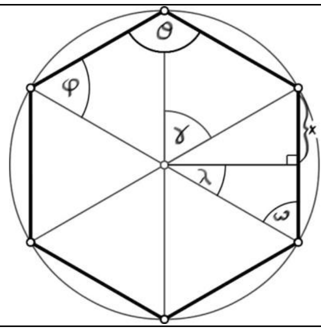
1: Come up with a generalized formula for finding the perimeter of any **<u>circumscribed</u> regular polygon**.

2: The Archimedean π approximation. Once you have a general formula for inscribed regular polygons and one for circumscribed regular polygons you can set about finding a value of π. Choose a polygon to try... say a square. Inscribe a square inside a circle of radius 1 and circumscribe a square around the same circle. Then find their perimeters of each square and π should be in-between. Try to get a value for π that is as close as you can by trying different matching polygons. What is the minimum n necessary to get a value of π accurate to 3 decimal places (i.e. 3.142)?

3: If you really want to be fancy, throw this stuff into Excel or some other computer application and print out some results.



SOLUTIONS TO OPTIONAL ASSIGNMENT BASIC GEO-TRIGONO-METRY

	Solution: Using the hexagon as a model on the left and a more generalized approach on the right.
Specifically a Hexagon	Generalization
The perimeter is six times one of the side-lengths. $6 \cdot (sidelength) = perimeter$	Generally the equation will be *n* times the length of a side: $n \cdot (side\ length) = perimeter,$ where n is the number of sides of a regular polygon.
The length of a side is twice the length of a half-side (Duh), labeled on the diagram as "*x*". We can determine *x* using trigonometry. $sin\lambda = \frac{x}{r} = sin30° = \frac{x}{1} =$ $sin30° = x = 0.5$	The length of *x*, a half-side, is $sin\lambda = \frac{x}{r}$. And r = 1. So, $sin\lambda = x.$ Alright. Fine. But how to write this in generalized terms. I'd like to input the number of sides, *n*, and get a perimeter as output. *P(n)*. So how do I do this?
That means that whole side is *2x = 2(0.5) = 1*.	Well, λ is pretty easy to write in terms of *n*. λ is half of γ and γ is just $\frac{360°}{n}$ and $\lambda = \frac{360°}{2n}$
$6 \cdot (sidelength) = perimeter$ ∴ The perimeter is 6.	So the perimeter of an *n*-sided regular polygon inscribed in a circle of radius 1 is... $Perimeter = n(2x)$ $= 2n(sin\lambda)$ $P_i(n) = 2n\left(sin\frac{360°}{2n}\right)$ $= 2n\left(sin\frac{180°}{n}\right)$ That's your answer.

Test drive it:

$$P(6) = 2(6)\left(sin\frac{180°}{6}\right)$$

$$= 12(sin30°)$$

$$= 12(0.5) = 6$$

Exactly what we hoped for.

So now, let's play with it. What's the ratio of perimeter to diameter? It's $6/2 = 3$. That's hardly a good approximation for π. So let's try a really big value for n. How about 100. A 100-sided regular polygon... is almost a circle... sort of... approximately.

$$P(100) = 2(100)\left(sin\frac{180°}{(100)}\right) \cong 6.28215$$

What's the ratio of perimeter to diameter for this 100-sided poly? It's $\frac{6.28215}{2} \cong 3.1412.$

Now that is starting to look like π. It differs from π by 0.000516745776... $\cong$ 5.2×10^{-4}.

$P(1000) \cong 3.141587$ [Differes from π by 0.000005167710... $\cong$ 5.2×10^{-6}]
$P(10000) \cong 3.141593$ [Differes from π by 0.000000051677... $\cong$ 5.2×10^{-8}]

Note: There is an interesting pattern showing up here in the differences between polygon-perimeter/diameter and π. That repeating "5167" is weird. This might be worth another look...or not. But this is a good example of how you follow your nose when exploring math. This type of mathematical curiosity is where Fields Medals come from.

Optional Homework solution: Circumscribed Regular Polygon Formula (for radius 1 circle)

$$P_c(n) = 2n\left(tan\frac{180°}{n}\right)$$

There are other ways to do this using other trigonometric functions, but this seemed the most obvious to me.

If you average circumscribed and inscribed polygons,
you reach an accuracy for π to 3 decimal points (3.142) at about n = 54.

Optional Addendum

The following table is a section from an Excel spreadsheet.
You can see where Excel punks out with it's approximation of π.
It is only accurate to 15 significant places.
In fact there is a glaring flaw for the very first measurement of perimeter for n = 1. I suspect the reason is that Excel demands angles to be measured in radians, not degrees.
And radians use π, and Excel's π is not very accurate.
Also... What is a polygon with 1 side?... or 2 for that matter?

Perimeter of regular polygon inscribed in a circle of radius 1.

π in Excel = 3.14159265358979000000000000000

General formula: P(n) = 2n(sin(180º/n))
P(n)=2*n*SIN(π/n)

Number of Sides	**Perimeter**	**Perimeter/Diameter**	**Difference from π**
n	**P(n)**	**P(n)/2**	**π – P(n)/2**
1	0.00000000000000024503	0.00000000000000012251	3.14159265358979000000000000000E+00
2	4.00000000000000000000	2.00000000000000000000	1.14159265358979000000000000000E+00
3	5.19615242270663000000	2.59807621135332000000	5.43516442236477000000000000000E-01
4	5.65685424949238000000	2.82842712474619000000	3.13165528843603000000000000000E-01
5	5.87785252292473000000	2.93892626146237000000	2.02666392127427000000000000000E-01
6	6.00000000000000000000	3.00000000000000000000	1.41592653589794000000000000000E-01
7	6.07437234764581000000	3.03718617382291000000	1.04406479766886000000000000000E-01
8	6.12293491784144000000	3.06146745892072000000	8.01251946690749000000000000000E-02
9	6.15636257986204000000	3.07818128993102000000	6.34113636587745000000000000000E-02
10	6.18033988749895000000	3.09016994374947000000	5.14227098403190000000000000000E-02
20	6.25737860160923000000	3.12868930080462000000	1.29033527851758000000000000000E-02
30	6.27170779605921000000	3.13585389802960000000	5.73875556018955000000000000000E-03
40	6.27672765822760000000	3.13836382911380000000	3.22882447599548000000000000000E-03
50	6.27905195293134000000	3.13952597646567000000	2.06667712412445000000000000000E-03
60	6.28031474915326000000	3.14015737457663000000	1.43527901316354000000000000000E-03
70	6.28107624907209000000	**3.14053812453604000000**	1.05452905374825000000000000000E-03
80	6.28157052145098000000	3.14078526072549000000	8.07392864304202000000000000000E-04
90	6.28190940645017000000	3.14095470322509000000	6.37950364705819000000000000000E-04
100	6.28215181562566000000	3.14107590781283000000	5.16745776963923000000000000000E-04
1,000	6.28317497175913000000	**3.14158748587956000000**	5.16771022951446000000000000000E-06
10,000	6.28318520382533000000	3.14159260191267000000	5.16771279102102000000000000000E-08
100,000	6.28318530614604000000	3.14159265307302000000	5.16771514469383000000000000000E-10
1,000,000	6.28318530716925000000	3.14159265358463000000	5.16742204581533000000000000000E-12
10,000,000	6.28318530717948000000	3.14159265358974000000	5.19584375524573000000000000000E-14
100,000,000	6.28318530717958000000	3.14159265358979000000	0.00000000000000000000000000000E+00
1,000,000,000	6.28318530717959000000	3.14159265358979000000	0.00000000000000000000000000000E+00
10,000,000,000	6.28318530717959000000	3.14159265358979000000	0.00000000000000000000000000000E+00

Stray observation:

$2x^2 - 2x - 2 = 0$ yields Golden Section

as does $2x^2 + 2x - 2 = 0$

Solve using Quadratic Formula